

THE GEOMETRY OF CIRCLES
ORTHOGONAL TO A GIVEN SPHERE

BY
CHARLES SAVAGE FORBES

SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY IN THE FACULTY OF PURE SCIENCE
COLUMBIA UNIVERSITY

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THE GEOMETRY OF CIRCLES ORTHOGONAL TO A GIVEN SPHERE

BY

CHARLES SAVAGE FORBES

INTRODUCTORY CONSIDERATIONS

It is well known that the progress of geometry in the past century has been due in large measure to recognition of the fact that the choice of element is arbitrary. To one trained in the Plücker line geometry, the conception of space as an assemblage of lines becomes both convenient and natural. We have chosen the circle as an element because in the domain to which we invite the reader's attention the circle is the natural element. We treat of the assemblage of circles orthogonal to a given sphere. At the outset we regard this system as a part of ordinary space, but once the foundations are laid we shall think and move in a space of circles.

Koenigs has roughly outlined the six dimensional theory of circles in space. Our space is related to his much as the plane is related to ordinary point space. We have restricted our domain, but have thereby rendered possible a more searching inquiry. The geometry of circles orthogonal to a given sphere is closely related to the Plücker line geometry. Both are four-dimensional, and like the geometry of the planes of a point in space of four dimensions each possesses a self-reciprocal element.

The circle is dually regarded as an envelope of spheres, or as a locus of point-pairs. The sphere and the point-pair are reciprocal notions, and the properties of configurations of circles are stated dually in terms of these fundamental concepts. Two sets of six circle coördinates arise, and a linear relation between either set defines a complex of circles. Point-pairs and spheres are correlated by each complex as pole and polar, and either contains all the circles of the complex belonging to the other. Hence the circles of a complex lying upon a sphere constitute a pencil of circles. Each circle has a conjugate circle with respect to every complex. Two complexes intersect in a congruence and three complexes in a circle surface, closely related to the hyperboloid of one sheet. The geometry of families of spheres is found similar to that of systems of parallel planes. The linear transformation of the spheres of reference, the transformation by inversion, and the projective transformations are utilized; and there is set up a geometry of polar and tangent linear complexes with respect to a complex of second degree.

I am indebted to Professor C. J. Keyser of Columbia University for the suggestion of the topic and much kindly criticism.

CHAPTER I

INTRODUCTORY THEOREMS UPON ORTHOGONAL SPHERES AND CIRCLES

In this chapter we establish certain elementary theorems upon which the subsequent theory is based.

THEOREM I. *There are ∞^3 planes in three dimensional space.*

The equation of a plane is

$$Ax + By + Cz + 1 = 0,$$

which involves three parameters, and gives to the plane three degrees of freedom.

THEOREM II. *There are ∞^6 circles in three dimensional space.*

The equation of a circle in a plane involves three parameters. No two planes have a circle of finite radius in common. Hence (Theorem I), there are

$$\infty^3 \times \infty^3 = \infty^6$$

circles in space.

THEOREM III. *There are ∞^4 spheres in three dimensional space.*

The equation of a sphere,

$$x^2 + y^2 + z^2 + 2ax + 2by + 2cz + d = 0$$

involves four parameters.

DEFINITION I. *Two spheres are orthogonal, when the square of the distance of their centers is equal to the sum of the squares of their radii.*

THEOREM IV. *There are ∞^3 spheres orthogonal to a given sphere.*

Definition I imposes one condition upon the four parameters of the arbitrary sphere, leaving to it three degrees of freedom.

DEFINITION II. *Two intersecting circles are orthogonal when their tangents at the point of intersection are perpendicular.*

DEFINITION III. *A circle is orthogonal to a sphere when it is orthogonal to every great circle of the sphere passing through one of the points of intersection.*

THEOREM V. *A circle orthogonal to two great circles of a sphere at their point of intersection is orthogonal to the sphere.*

For the tangents to the great circles through the point of intersection all lie in a plane tangent to the sphere at the point. The tangent of the circle being perpendicular to two of these tangent lines is perpendicular to the plane and therefore to them all. Furthermore this tangent line being perpendicular to a

tangent plane at the point of tangency goes through the center of the sphere. A circle and its tangent line lie in a same plane. From this follows:

THEOREM VI. *Every circle orthogonal to a sphere, lies in a plane passing through the center of the sphere.*

THEOREM VII. *Every sphere passing through a circle orthogonal to a sphere is orthogonal to that sphere.*

Let C be a circle orthogonal to a sphere S . Pass a plane Π through C and point O the centre of S (Theorem VI). Π cuts out of S a circle C' , orthogonal to circle C (Definition III). Let A be the centre of circle C . Let P be one of the points of intersection of circles C and C' . Draw the tangents OP and AP to C and C' . These tangents are perpendicular. Pass any sphere S' through circle C . The center O' of S' is in a perpendicular AO' to the plane Π . $O'P$ is therefore perpendicular to OP . Hence

$$\overline{OO'}^2 = \overline{OP}^2 + \overline{O'P}^2.$$

But OO' is the distance of the centres of spheres S and S' ; and OP and $O'P$ are their respective radii. Therefore S and S' are orthogonal (Definition I).

We next find the analytical condition (rectangular coördinates) that the spheres

$$(S) \quad x^2 + y^2 + z^2 = R^2,$$

$$(S') \quad x^2 + y^2 + z^2 + 2ax + 2by + 2cz + a^2 + b^2 + c^2 = r^2,$$

shall be orthogonal. The condition is (Definition I)

$$a^2 + b^2 + c^2 = R^2 + r^2.$$

Whence

$$r^2 = a^2 + b^2 + c^2 - R^2$$

and S' becomes

$$x^2 + y^2 + z^2 + 2ax + 2by + 2cz = -R^2.$$

Hence the result:

THEOREM VIII. *A sphere S' is orthogonal to a sphere S , whose center is the origin, when the absolute term of S' is equal to the square of the radius of S .*

THEOREM IX. *Every plane passing through the center of a sphere S , cuts out of a second sphere S' , orthogonal to S , a circle C' , orthogonal to the circle C cut out of S by the plane.*

Let the equations of the spheres be

$$(S) \quad x^2 + y^2 + z^2 = R^2,$$

$$(S') \quad x^2 + y^2 + z^2 + 2ax + 2by + 2cz = -R^2.$$

Since the property is geometrical, it is independent of the choice of axes. The cutting plane may therefore be taken as

$$z = 0.$$

This gives the circles

$$(C) \quad x^2 + y^2 = R^2,$$

$$(C'') \quad x^2 + y^2 + 2ax + 2by = -R^2.$$

These circles are orthogonal, for they lie in a plane, and

$$a^2 + b^2 \equiv R^2 + a^2 + b^2 - R^2.$$

That is, the square of the distance of their centers is equal to the sum of the squares of their radii. Hence their tangents at the points of intersection are perpendicular, and they are orthogonal (Definition II).

THEOREM X. *Two spheres orthogonal to a same sphere intersect in a circle orthogonal to that sphere.*

Let the spheres

$$(S') \quad x^2 + y^2 + z^2 + 2ax + 2by + 2cz = -R^2,$$

$$(S'') \quad x^2 + y^2 + z^2 + 2a'x + 2b'y + 2c'z = -R^2,$$

be orthogonal to

$$(S) \quad x^2 + y^2 + z^2 = R^2.$$

The intersection of S' and S'' is a circle C lying in the plane

$$(II) \quad (a - a')x + (b - b')y + (c - c')z = 0.$$

This plane Π passes through the center, O , of sphere S . Let P be one of the points where circle C intersects sphere S . Pass a plane Π' through P and O , perpendicular to plane Π . Π' cuts out of sphere S a circle C' orthogonal to C (Definition II). We have the Lemma:

Every circle lying in a plane which passes through the center of a sphere is orthogonal to a great circle of that sphere at each of its points of intersection with the sphere.

C is also orthogonal to circle C'' cut out of S by Π (Theorem IX). C being orthogonal to two great circles C' and C'' of sphere S , is orthogonal to S (Theorem V).

Theorem X may be analytically established as follows, whether the circle of intersection be real or imaginary. The equation of the fundamental sphere is

$$(S) \quad x^2 + y^2 + z^2 = R^2.$$

Any two spheres orthogonal to S are given by

$$(S') \quad x^2 + y^2 + z^2 + 2a'x + 2b'y + 2c'z + R^2 = 0,$$

$$(S'') \quad x^2 + y^2 + z^2 + 2(a + a')x + 2(b + b')y + 2(c + c')z + R^2 = 0.$$

The axial plane of S' and S'' is

$$(\Pi) \quad ax + by + cz = 0,$$

and of S and S' is

$$(\Pi') \quad a'x + b'y + c'z + R^2 = 0.$$

The intersections of Π , Π' and S are the points of intersection of the three spheres, i. e., the points where the circle (S', S'') pierces S . Solving simultaneously for x' , y' and z' the coördinates of these points we obtain,

$$\begin{aligned} x' &= \frac{[c(c'a - ca') + b(b'a - ba')]}{F'_s} R^2 \pm R(c'b - cb') \sqrt{F'_s}, \\ y' &= \frac{[a(a'b - ab') + c(c'b - cb')]}{F'_s} R^2 \pm R(a'c - ac') \sqrt{F'_s}, \\ z' &= \frac{[a(a'c - ac') + b(b'c - bc')]}{F'_s} R^2 \pm R(b'a - a'b) \sqrt{F'_s}, \end{aligned}$$

where

$$F'_s = (ac' - a'c)^2 + (bc' - b'c)^2 + (ab' - a'b)^2 - R^2(a^2 + b^2 + c^2),$$

and

$$F'_s = (ac' - a'c)^2 + (bc' - b'c)^2 + (a'b - ab')^2.$$

It is observed that F'_s and F'_s are symmetric in a, b, c, a', b', c' ; that x', y', z' are finite (for finite a, b , etc.) unless $a : b : c = a' : b' : c'$; and that x', y', z' are real or imaginary according as $F'_s \geq 0$, or $F'_s < 0$.

The tangent plane to S at $(x' y' z')$ is

$$\frac{\partial S}{\partial x'}(x - x') + \frac{\partial S}{\partial y'}(y - y') + \frac{\partial S}{\partial z'}(z - z') = 0,$$

or since

$$x'^2 + y'^2 + z'^2 = R^2,$$

$$(\Pi'') \quad xx' + yy' + zz' = R^2.$$

This plane contains the tangent lines of all great circles of S , passing through $(x' y' z')$. To prove a circle orthogonal to S , it is only necessary, therefore, to prove the tangent to the circle at the point $(x' y' z')$ perpendicular to this tangent plane at $(x' y' z')$. The plane is real only when $F'_s \geq 0$. The tangent to the circle of intersection of S' and S'' at $(x' y' z')$ is the intersection of plane

Π and the tangent plane of one of the spheres, say S' , at the point. This tangent plane is

$$\frac{\partial S'}{\partial x'}(x - x') + \frac{\partial S'}{\partial y'}(y - y') + \frac{\partial S'}{\partial z'}(z - z') = 0.$$

or

$$(x' + a')(x - x') + (y' + b')(y - y') + (z' + c')(z - z') = 0.$$

We have the relations

$$x'^2 + y'^2 + z'^2 = R^2,$$

$$a'x' + b'y' + c'z' = -R^2.$$

Hence the above reduces to

$$(x' + a')x + (y' + b')y + (z' + c')z = 0.$$

Since both this plane and plane Π pass through the origin and $(x'y'z')$, their line of intersection does the same, and is perpendicular to the plane Π'' , tangent to S . But this line is the tangent of the circle (S', S'') at the point of intersection with S . Hence circle (S', S'') is orthogonal to every great circle of S passing through $(x'y'z')$ and is orthogonal to S . At no point in the argument have we assumed that $(x'y'z')$ was a real point. Hence Theorem X holds for both the real and imaginary cases.

THEOREM XI. *Every circle orthogonal to a sphere at one point of intersection is orthogonal at the other also.*

This follows at once from the fact that the circle lies in a plane passing through the center of the sphere (Theorem VI), and from the symmetry of the construction.

THEOREM XII. *There are ∞^4 circles orthogonal to a given sphere.*

Theorem V imposes two conditions upon the six parameters of the circle, leaving four degrees of freedom. Again there are ∞^2 points on the sphere. Through each point P and the center of the sphere, ∞ planes may be passed. In each of these planes ∞ circles may be drawn through P , orthogonal to the intersection of the plane and the sphere, and hence orthogonal to the sphere (Definition III; also see proof of Theorem X). Therefore through any point on the sphere, ∞^2 circles may be passed orthogonal to the sphere. Hence, since each circle cuts the sphere in a finite number (2) of points, there are

$$\infty^2 \times \infty^2 = \infty^4$$

circles orthogonal to a given sphere.

THEOREM XIII. *Two points on the sphere fix one of the orthogonal circles. Pass a plane through the points P and P' , and the centre of the sphere. In*

this plane one and but one circle may be passed through P and P' , and orthogonal to the circle formed by the intersection of the plane and the sphere.

THEOREM XIV. *All those circles orthogonal to the same sphere, which have one point in common, have a second point in common related to the first point by inversion with respect to the sphere.*

Let the circles C' and C'' be orthogonal to a sphere S , and intersect in a point P . P may be taken without the sphere. Let S' be any sphere passing through C' but not through C'' . Spheres S and S' are orthogonal (Theorem VII). Circle C'' lies in a plane Π drawn through P and O , the centre of S (Theorem VI). This plane cuts out of sphere S' a circle C''' orthogonal to sphere S (Theorem XI; Lemma, Theorem X). Draw OP . Plane Π cuts out of S a circle C . C'' and C''' therefore lie in the same plane Π with circle C and are orthogonal to it. C''' cuts OP a second time at B . Therefore by the theory of orthogonal pencils of circles* C'' also passes through B . That is P and B are reciprocal poles with respect to the circle C . Circle C' lies in the same plane with OP and therefore cuts OP a second time. But C' lies on sphere S' and therefore must cut OP in the point in which OP cuts S' . This point is B since C''' lies on S' and passes through B . Hence C' and C'' intersect a second time in B .

We next establish that the analytical relation between the points P and B is that of inversion with respect to the sphere S . Assume O , the center of S , as the origin. The equation of S is

$$x^2 + y^2 + z^2 = R^2.$$

Since the configuration is independent of a rotation of axes, the plane Π may be assumed as

$$z = 0.$$

The circle C cut from S by Π is

$$x^2 + y^2 = R^2.$$

Let OP be the axis of X and P be the point $(K \ 0 \ 0)$. Since the circle C''' cut from S' by Π , is orthogonal to C , its equation is of the form

$$x^2 + y^2 - 2ax - 2by = -R^2.$$

Since C''' passes through the point $(K \ 0 \ 0)$,

$$K^2 - 2aK = -R^2,$$

i. e.,

$$a = \frac{K^2 + R^2}{2K}.$$

* KLEIN, *Einleitung in die höhere Geometrie*, vol. I, p. 87.

The equation of C''' is therefore

$$x^2 - \frac{K^2 + R^2}{K}x + y^2 - 2by = -R^2.$$

This circle cuts the axis of X in the points (P and B)

$$(K \ 0 \ 0) \quad \text{and} \quad (R^2/K \ 0 \ 0).$$

Since the coördinates of B do not involve the coefficient b of the equation defining C''' , it is seen that P and B are uniquely related, i. e., since any two circles orthogonal to S and intersecting in P , also intersect in B , *all such circles passing through P will also pass through B* . The coördinates of P and B satisfy the formulæ of inversion

$$x'' = \frac{x' R^2}{x'^2 + y'^2}, \quad y'' = \frac{y' R^2}{x'^2 + y'^2},$$

and are therefore reciprocal poles with respect to the circle C .

Since the configuration is independent of a rotation of axes, the relation between P and B is unaltered by a rotation of plane Π about OP as axis. P and B are therefore reciprocal poles with respect to the sphere, and their coördinates satisfy the formulæ of inversion

$$x'' = \frac{x' R^2}{x'^2 + y'^2 + z'^2}, \quad y'' = \frac{y' R^2}{x'^2 + y'^2 + z'^2}, \quad z'' = \frac{z' R^2}{x'^2 + y'^2 + z'^2}.$$

For it has been shown that if $\overline{OP} = K$, then $\overline{OB} = R^2/K$. That is $\overline{OP} \times \overline{OB} = R^2$. Now let P be the point $(x' y' z')$. The point $(x'' y'' z'')$ lies upon OP . For the equation of the line OP is

$$xy'z' = yx'z' = zx'y'.$$

Substituting the values of the coördinates of $(x'' y'' z'')$, we have the identity

$$\frac{x' y' z' R^2}{x'^2 + y'^2 + z'^2} = \frac{x' y' z' R^2}{x'^2 + y'^2 + z'^2} = \frac{x' y' z' R^2}{x'^2 + y'^2 + z'^2}.$$

Furthermore the distance

$$\overline{OP} = \sqrt{x'^2 + y'^2 + z'^2};$$

and if $(x'' y'' z'')$ is denoted by B ,

$$\overline{OB} = \frac{\sqrt{(x'^2 + y'^2 + z'^2)} R^2}{x'^2 + y'^2 + z'^2}.$$

$$\therefore \quad \overline{OP} \times \overline{OB} = R^2.$$

Hence to every point without a sphere S there corresponds a point within. These two points are related by the formulæ of inversion. If two circles orthogonal to the sphere S pass through a same point without the sphere, they pass through the corresponding point within. Two circles orthogonal to a same sphere, therefore intersect in two points or not at all. No two such circles can intersect in two points without or within the sphere, else they intersect in two other points corresponding to these two, and therefore coincide. It is readily seen that the relationship between corresponding points is reciprocal. In other words throughout the argument the terms point without and point within may be interchanged.

Particular attention is called to Theorems X, XII and XIV, which are of especial consequence in the subsequent chapters. We repeat these theorems for the convenience of the reader.

THEOREM X. *The intersection of two spheres orthogonal to a given sphere is a circle orthogonal to that sphere.*

THEOREM XII. *There are ∞^4 circles orthogonal to a given sphere.*

THEOREM XIV. *All those circles orthogonal to a same sphere, which intersect in one point, intersect in a second point the reciprocal pole of the first point with respect to the sphere.*

It is noted in passing that Theorem X is but a generalization of the theorem: *The intersection of two planes perpendicular to a third plane is a line perpendicular to the third plane.*

We pass from these preliminary investigations to the main body of our paper, and shall henceforth make exclusive use of systems of circle-coördinates derived from the Darboux pentaspherical coördinates.

CHAPTER II

COÖRDINATES OF THE CIRCLE

A. Pentaspherical Coördinates

1. Choose five mutually orthogonal spheres

$$S_1, S_2, S_3, S_4, S_5.$$

Let their radii be

$$R_1, R_2, R_3, R_4, R_5.$$

Any point is determined by five Darboux* pentaspherical coördinates,

$$x_1, x_2, x_3, x_4, x_5;$$

where

$$x_i = \rho \frac{P_i}{R_i} \quad (i=1, 2, \dots, 5);$$

* DARBOUX, *Théorie des Surfaces*, vol. 1, book II, chap. 6.

P_i denoting the power of the point with respect to the sphere S_i , and ρ being an arbitrary factor. The five coördinates are homogeneous, and are connected by the identity

$$(1) \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \equiv 0.$$

Any equation of the form

$$(2) \quad a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 + a_5 x_5 \equiv \sum_1^5 a_i x_i = 0,$$

where the quantities a_i are constants, represents a sphere.* The equations of the five fundamental spheres of reference, S_i , are

$$(3) \quad x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0, \quad x_5 = 0.$$

These spheres intersect in ten circles

$$(4) \quad x_i = x_j = 0 \quad (i, j = 1, \dots, 5; i \neq j);$$

and in ten pairs of points

$$(5) \quad x_i = x_j = x_k = 0 \quad (i, j, k = 1, \dots, 5; i \neq j \neq k).$$

2. The condition that two spheres

$$(6) \quad \sum_1^5 a_i x_i = 0, \quad \sum_1^5 b_i x_i = 0$$

shall be orthogonal is

$$(7) \quad a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 + a_5 b_5 = 0.$$

3. We are considering the geometry of circles orthogonal to a given sphere. The choice of the five fundamental spheres, S_i , allows that the given sphere be taken as any one of the five. We choose to take it as S_5 , and to give it a real positive† radius. Although this latter restriction is of no advantage algebraically, it serves to clarify the geometrical conceptions.

4. The equation of a sphere orthogonal to S_5 , is of the form

$$(8) \quad a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 \equiv \sum a_i x_i = 0.$$

[See (3) and (7).]

There are ∞^3 such spheres, since (8) contains three parameters (Cf. Theorem IV). Any two such spheres intersect in a circle, real or imaginary, orthogonal to S_5 . This has been proved in Theorem X for both the case where the circle of intersection is real, and where it is imaginary. There are ∞^4 circles orthogonal to S_5 (Theorem XII).

* Cf. KOENIGS, *Contributions à la théorie du cercle dans l'espace*, Annales ... Toulouse, 1888, p. 1; DARBOUX, op. cit.

† One of the five spheres must have a negative radius. See KLEIN, *Einleitung*, vol. 1, p. 102.

Although there are six parameters involved in the equations of the two spheres intersecting to form the circle, the latter can be thus generated in ∞^2 ways. All the circles orthogonal to S_5 may be generated by the intersections of the spheres orthogonal to S_5 .

5. Since we treat exclusively of the spheres and circles orthogonal to S_5 , we shall use the terms *sphere* and *circle* to mean respectively, *sphere orthogonal to S_5* , and *circle orthogonal to S_5* .

The configuration of reference consists therefore of the four mutually orthogonal spheres

$$S_1, S_2, S_3, S_4, \quad \text{i. e.,} \quad x_i = 0 \quad (i=1, \dots, 4).$$

These spheres intersect in six circles

$$x_i = x_j = 0 \quad (i, j=1, \dots, 4; i \neq j),$$

and in four pairs of points,

$$x_i = x_j = x_k = 0 \quad (i, j, k=1, \dots, 4; i \neq j \neq k).$$

Attention is called to the similarity between this *tetrasphere* of reference and the tetrahedron of reference in line geometry. The following notions correspond.

TETRAHEDRON		TETRASPHERE
4 faces	planes	spheres
6 edges	lines	circles
4 vertices	points	pairs of points

It is noted that the four spheres of reference belong to the assemblage of spheres orthogonal to S_5 , but that S_5 itself does not. The coördinates of a point referred to this configuration are tetraspherical.

B. Point-pairs

6. Three spheres of the assemblage intersect in two points, one within and one without S_5 (Theorem XIV). All those circles (or spheres) which pass through one of the points pass through the other also. Conversely, if two circles of the assemblage intersect in two points, these points are related by inversion. We denote two points so related as a *point-pair*, or simply as a *point*.

The pentaspherical coördinates of the point-pair may arbitrarily be taken as those of the outer element, but we now show that inasmuch as a point is fixed by the ratios $x_1 : x_2 : x_3 : x_4$, it makes no difference whether the inner or outer element is selected.

Let $(x'y'z')$ and $(x''y''z'')$ be the elements of a point-pair. We have by the formulæ of inversion with respect to the fundamental sphere

$$(S_5) \quad x^2 + y^2 + z^2 = R^2,$$

the relations

$$x'' = \frac{x' R^2}{x'^2 + y'^2 + z'^2}, \quad y'' = \frac{y' R^2}{x'^2 + y'^2 + z'^2}, \quad z'' = \frac{z' R^2}{x'^2 + y'^2 + z'^2}.$$

Let S_i , one of the fundamental spheres, not S_5 , be

$$x^2 + y^2 + z^2 + 2ax + 2by + 2cz + R^2 = 0.$$

S_i is orthogonal to S_5 .

With reference to $(x'y'z')$

$$x'_i = \rho \frac{P_i}{R_i} = \rho \frac{x'^2 + y'^2 + z'^2 + 2ax' + 2by' + 2cz' + R^2}{R_i} \quad (i=1, \dots, 4; \rho \text{ arbitrary}).$$

With reference to $(x''y''z'')$

$$\begin{aligned} x''_i &= \rho \frac{\frac{(x'^2 + y'^2 + z'^2)}{(x'^2 + y'^2 + z'^2)^2} R^4 + \frac{2ax' + 2by' + 2cz'}{x'^2 + y'^2 + z'^2} R^2 + R^2}{R_i} \\ &= \rho \frac{R^2}{x'^2 + y'^2 + z'^2} \cdot \frac{x'^2 + y'^2 + z'^2 + 2ax' + 2by' + 2cz' + R^2}{R_i}. \end{aligned}$$

Hence we have the relations

$$(9) \quad x''_i = \frac{R^2}{x'^2 + y'^2 + z'^2} x'_i \quad (i=1, \dots, 4).$$

$$x'_5 = \rho \frac{P_5}{R} = \rho \frac{x'^2 + y'^2 + z'^2 - R^2}{R},$$

$$x''_5 = \rho \frac{\frac{x'^2 + y'^2 + z'^2}{(x'^2 + y'^2 + z'^2)^2} R^4 - R^2}{R} = -\rho \frac{R^2}{x'^2 + y'^2 + z'^2} \cdot \frac{x'^2 + y'^2 + z'^2 - R^2}{R},$$

i. e.,

$$(10) \quad x''_5 = -\frac{R^2}{x'^2 + y'^2 + z'^2} x'_5.$$

The theorem follows:

The pentaspherical coördinates of the elements of a point-pair are respectively proportional, save that the coördinates x_5 differ in sign, i. e., $x_1 : x_2 : x_3 : x_4 : x_5 = x'_1 : x'_2 : x'_3 : x'_4 : -x'_5$.

The coördinate x_5 of the inner point is always negative.* For if the inner point is (abc)

$$P_5 = a^2 + b^2 + c^2 - R^2 < 0, \quad \therefore x_5 = \rho \frac{P_5}{R} < 0.$$

* For ρ positive.

Therefore to every point-pair there is a definite set of ratios $x_1 : x_2 : x_3 : x_4$, and conversely, to every set of these ratios, there is a definite point-pair. The coördinate x_5 may be determined from the identity $\sum_1^5 x_i^2 = 0$, the sign being $\pm$ according as the $\left\{ \begin{smallmatrix} \text{outer} \\ \text{inner} \end{smallmatrix} \right\}$ element of the point-pair is taken.

CORRESPONDENCE I. To every point without a sphere S_5 corresponds a unique point within, and reciprocally.

To the center of the sphere, however, corresponds the plane at infinity and reciprocally. If the radius of the sphere is zero, the sphere is a point which corresponds to every point of space.

$$C. \text{ Dual Interpretation of } \sum_1^4 a_i x_i = 0$$

7. The equation $\sum_1^4 a_i x_i = 0$, may be interpreted in a dual sense. If we regard the quantities a_i as constant, and the quantities x_i as variable, equation (8) represents a sphere as the locus of its points. By virtue of the identity (1), a point is known when four of its coördinates are given. (Cf. Section 6.) Three points

$$x'_1, \dots, x'_4, \quad x''_1, \dots, x''_4, \quad x'''_1, \dots, x'''_4,$$

determine the sphere, since they determine the ratios of the quantities a_i uniquely, provided the points are not on a same circle, i. e.,

$$(11) \quad \begin{vmatrix} x'_1 & x'_2 & x'_3 & x'_4 \\ x''_1 & x''_2 & x''_3 & x''_4 \\ x'''_1 & x'''_2 & x'''_3 & x'''_4 \end{vmatrix} \neq 0.$$

The same sphere is equally well determined by any three points, whose coördinates are of the form

$$(12) \quad x_i = \lambda_1 x'_i + \lambda_2 x''_i + \lambda_3 x'''_i \quad (i=1, \dots, 4; \lambda_1, \lambda_2, \lambda_3 \text{ arbitrary}).$$

Conversely every point whose coördinates are of the form (12) is on the sphere.*

Reciprocally if the quantities x_i are regarded as fixed, equation (8) represents a point-pair x_i as the envelope of the spheres which pass through it. Three sets of sphere coördinates

$$a'_1, \dots, a'_4, \quad a''_1, \dots, a''_4, \quad a'''_1, \dots, a'''_4$$

fix the point, provided the three spheres do not pass through a same circle, i. e.,

$$(13) \quad \begin{vmatrix} a'_1 & a'_2 & a'_3 & a'_4 \\ a''_1 & a''_2 & a''_3 & a''_4 \\ a'''_1 & a'''_2 & a'''_3 & a'''_4 \end{vmatrix} \neq 0.$$

* Cf. KEYSER, *Plane Geometry of the Point in Space of Four Dimensions*, pp. 311-313; KÖNIGS, *La Géométrie Réglée*, Annales-Toulouse, 1889, p. 11, et al.

The point is equally well determined by any three spheres whose coördinates are of the form

$$(14) \quad a_i = \lambda_1 a'_i + \lambda_2 a''_i + \lambda_3 a'''_i \quad (i=1, \dots, 4; \lambda_1, \lambda_2, \lambda_3 \text{ arbitrary}).$$

Conversely every sphere whose coördinates are of the form (14), passes through the point.

D. Coördinates of Circle as Envelope of Spheres

8. All the spheres passing through a given circle constitute a *pencil of spheres*. If the circle is given by the intersection of the spheres

$$\sum a_i x_i = 0, \quad \sum b_i x_i = 0$$

the coördinates of the spheres of the pencil are of the form

$$a_i = \lambda_1 a_i + \lambda_2 b_i \quad (\lambda_1, \lambda_2 \text{ arbitrary}).$$

The circle is equally well defined by any two spheres of the pencil

$$\sum (a_i + \lambda b_i) x_i = 0 \quad (\lambda = \lambda_2 / \lambda_1).$$

9. In particular it is defined by any two of the four spheres of the pencil, which are orthogonal, each to one of the four fundamental spheres

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0.$$

The equations of these four spheres of the pencil are

$$(15) \quad \begin{cases} * & + p_{12}x_2 + p_{13}x_3 + p_{14}x_4 = 0, \\ p_{21}x_1 + * & + p_{23}x_3 + p_{24}x_4 = 0, \\ p_{31}x_1 + p_{32}x_2 + * & + p_{34}x_4 = 0, \\ p_{41}x_1 + p_{42}x_2 + p_{43}x_3 + * & = 0, \end{cases}$$

where

$$p_{ik} = a_i b_k - a_k b_i \quad (i, k = 1, \dots, 4).$$

Obviously

$$p_{ii} = 0; \quad p_{ik} = -p_{ki}.$$

Expanding the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix} \equiv 0,$$

* Cf. KOENIGS, *La Géométrie Réglée*, Annales... Toulouse, 1889, p. 6.

in terms of its second order minors, we obtain

$$(16) \quad \omega(p) \equiv p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} \equiv 0.$$

Although two of equations (15) are sufficient to determine the circle, all four are retained for the sake of symmetry, and the *six* * *coördinates* p_{ik} are taken as the homogeneous coördinates of a circle orthogonal to the fundamental sphere S_5 . The identity (16) reduces the degree of freedom of the system to four as is required by Theorem XII.

E. Coördinates of Circle as Locus of Point-Pairs

10. Reciprocally two point-pairs

$$\sum a_i x_i = 0, \quad \sum a_i y_i = 0$$

determine a circle, and the same circle is determined by any two points of the range

$$\sum (x_i + \lambda y_i) a_i = 0 \quad (\lambda \text{ arbitrary}).$$

(By range of points is meant all the point-pairs on a circle which contains the generators of the range.) In particular, the circle is determined by any two of the four points of the range lying on the fundamental spheres. The equations of these points are

$$(17) \quad \begin{cases} * + q_{12}a_2 + q_{13}a_3 + q_{14}a_4 = 0, \\ q_{21}a_1 + * + q_{23}a_3 + q_{24}a_4 = 0, \\ q_{31}a_1 + q_{32}a_2 + * + q_{34}a_4 = 0, \\ q_{41}a_1 + q_{42}a_2 + q_{43}a_3 + * = 0, \end{cases}$$

where

$$q_{ik} = x_i y_k - x_k y_i, \quad (i, k = 1, \dots, 4).$$

$$q_{ii} = 0, \quad q_{ik} = -q_{ki}.$$

The identity exists

$$(18) \quad \omega(q) \equiv q_{12}q_{34} + q_{13}q_{42} + q_{14}q_{23} = 0.$$

11. As shown in the precisely similar algebra of line geometry,† if p_{ik} and q_{ik} are the coördinates of a same circle,

$$(19) \quad \frac{p_{12}}{q_{34}} = \frac{p_{13}}{q_{42}} = \frac{p_{14}}{q_{23}} = \frac{p_{34}}{q_{12}} = \frac{p_{42}}{q_{13}} = \frac{p_{23}}{q_{14}}.$$

* Cf. PLÜCKER, *Neue Geometrie des Raumes*, p. 2; CAYLEY, *Six Coördinates of a Line*, *Collected Works*, vol. 7, p. 66.

† PLÜCKER, *Neue Geometrie*, p. 5. KOENIGS, *La Géométrie Réglée*, *Annales*...Toulouse; 1889, p. 8.

Hence six quantities γ_{ik} , such that

$$\gamma_{12} = \rho p_{12} = \sigma q_{34}, \quad \gamma_{13} = \rho p_{13} = \sigma q_{42}, \text{ etc.} \quad (\rho, \sigma \text{ arbitrary}),$$

may be assumed as the homogeneous coördinates of a circle, without regard to the method of generation, or if the coördinates are taken in a certain order they give the circle as an envelope of spheres, or if the same coördinates are taken in a different order the circle is regarded as the locus of its points (i. e., point-pairs). Conversely, given six quantities γ_{ik} , which satisfy the identity

$$\omega(\gamma) = 0,$$

they may be taken as the coördinates of a circle, and be interpreted in a dual sense according to the order in which they are taken.

F. Conditions that Two Circles may Intersect

12. Two circles intersect* when their generating spheres (points) have a point (sphere) in common. Let the given circles be

$$(20) \quad \begin{cases} * + \gamma_{12}x_2 + \gamma_{13}x_3 + \gamma_{14}x_4 = 0, \\ \gamma_{21}x_1 + * + \gamma_{23}x_3 + \gamma_{24}x_4 = 0, \end{cases}$$

$$(21) \quad \begin{cases} * + \gamma'_{12}x_2 + \gamma'_{13}x_3 + \gamma'_{14}x_4 = 0, \\ \gamma'_{21}x_1 + * + \gamma'_{23}x_3 + \gamma'_{24}x_4 = 0. \end{cases}$$

These four spheres (points) have a point (sphere) in common, provided

$$(22) \quad \begin{vmatrix} 0 & \gamma_{12} & \gamma_{13} & \gamma_{14} \\ \gamma_{21} & 0 & \gamma_{23} & \gamma_{24} \\ 0 & \gamma'_{12} & \gamma'_{13} & \gamma'_{14} \\ \gamma'_{21} & 0 & \gamma'_{23} & \gamma'_{24} \end{vmatrix} = 0.$$

By the aid of

$$\gamma_{ik} = -\gamma_{ki}$$

and

$$\omega(\gamma) = 0$$

this reduces to

$$(23) \quad \omega(\gamma, \gamma') = \gamma_{12}\gamma'_{34} + \gamma_{13}\gamma'_{42} + \gamma_{14}\gamma'_{23} + \gamma'_{12}\gamma_{34} + \gamma'_{13}\gamma_{42} + \gamma'_{14}\gamma_{23} = 0.$$

This is therefore the condition that the circles γ_{ik} and γ'_{ik} shall intersect in a point-pair, or dually shall lie upon a same sphere.

*Cf. KOENIGS, *La Géométrie Régliée*, op. cit., p. 9.

G. Conditions that a Given Circle may lie on a Given Sphere or pass through a Given Point-pair

13. We seek the condition that a given circle (envelope of spheres) may lie upon a given sphere. Let the sphere be given by

$$(24) \quad c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0;$$

and the circle by

$$(25) \quad \begin{cases} a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0, \\ b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 = 0. \end{cases}$$

The condition is that the sphere is one of the spheres of the pencil of spheres, which is defined by (25), i. e.,

$$c_i = \lambda_1 a_i + \lambda_2 b_i \quad (\lambda_1, \lambda_2 \text{ arbitrary}),$$

This involves the two conditions

$$(26) \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} = 0.$$

14. Reciprocally, the condition that a circle (locus of points) given by two points shall pass through a third point, is that the point shall belong to the range defined by the two points which give the circle.

Let the point be

$$(27) \quad a_1z_1 + a_2z_2 + a_3z_3 + a_4z_4 = 0,$$

and the circle be

$$(28) \quad \begin{cases} a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0, \\ a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4 = 0. \end{cases}$$

The condition is

$$z_i = \lambda_1 x_i + \lambda_2 y_i \quad (\lambda_1, \lambda_2 \text{ arbitrary}).$$

This involves the two conditions

$$(29) \quad \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} = 0.$$

15. The condition that a given circle locus of points shall lie on a given sphere, is that both points which define the circle shall lie on the sphere. If the sphere be

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0,$$

and the points be y_i and z_i , the conditions are

$$(30) \quad \begin{cases} a_1 y_1 + a_2 y_2 + a_3 y_3 + a_4 y_4 = 0, \\ a_1 z_1 + a_2 z_2 + a_3 z_3 + a_4 z_4 = 0. \end{cases}$$

16. Reciprocally, the condition that a given circle, envelope of spheres shall pass through a given point, is that both spheres which define the circle shall pass through the point. If the point is

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = 0,$$

and the circle is defined by the spheres b_i and c_i , the conditions are

$$(31) \quad \begin{cases} b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4 = 0, \\ c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 = 0. \end{cases}$$

17. The last four sections exhibit the similarity between the reciprocal algebras of the circle regarded as an envelope of spheres and as a locus of points. The necessary and sufficient conditions for the union of the circle and sphere, and the circle and point in the one theory, are algebraically identical with the necessary and sufficient conditions for the union of the circle and point, and the circle and sphere in the reciprocal theory.

CHAPTER III

THE LINEAR COMPLEX OF CIRCLES

18. Equations of relation between the coördinates of the circle serve to select special systems of circles from the ∞^4 circles of the space. The system defined by a single equation is called a *complex*. If the equation be linear, the complex defined by it is called *linear*.

A. Linear Complex of Circles Regarded as Envelopes of Spheres

19. The equation of a linear complex of circles regarded as envelopes of spheres may be written *

$$(32) \quad A_{12} p_{12} + A_{13} p_{13} + A_{14} p_{14} + A_{23} p_{23} + A_{42} p_{42} + A_{34} p_{34} = 0.$$

Expanding, we obtain

$$(33) \quad \begin{aligned} & A_{12}(a_1 b_2 - a_2 b_1) + A_{13}(a_1 b_3 - a_3 b_1) \\ & + A_{14}(a_1 b_4 - a_4 b_1) + A_{23}(a_2 b_3 - a_3 b_2) \\ & + A_{42}(a_4 b_2 - a_2 b_4) + A_{34}(a_3 b_4 - a_4 b_3) = 0. \end{aligned}$$

* Cf. PLÜCKER, *Neue Geometrie*, p. 27.

Rearranging according to terms in a_i , we obtain

$$\begin{aligned}
 (34) \quad & (+A_{12}b_2 + A_{13}b_3 + A_{14}b_4)a_1 \\
 & + (-A_{12}b_1 + A_{23}b_3 - A_{42}b_4)a_2 \\
 & + (-A_{13}b_1 - A_{23}b_2 + A_{34}b_4)a_3 \\
 & + (-A_{14}b_1 + A_{42}b_2 - A_{34}b_3)a_4 = 0.
 \end{aligned}$$

If the quantities b_i are fixed, they define a sphere

$$(35) \quad \sum b_i x_i = 0.$$

If we regard the quantities a_i as variable, equation (34) is the equation of a point, the envelope of all the spheres of the circles of the complex, lying upon the sphere (35); in other words all circles of the complex lying upon the sphere (35) pass through the point (34). Hence, *to each sphere of space corresponds with respect to each complex one point.* The point lies upon the sphere since

$$(36) \quad a_i = b_i$$

satisfies equation (34). Therefore

All the circles of the complex lying on a sphere pass through a point of the sphere and form a pencil of circles.

B. Complex of Circles Regarded as Loci of Point-Pairs

20. Reciprocally, the equation of the complex may be written

$$(37) \quad A_{12}q_{34} + A_{13}q_{42} + A_{14}q_{23} + A_{23}q_{14} + A_{42}q_{13} + A_{34}q_{12} = 0.$$

[See (19).]

Expanding, we obtain

$$\begin{aligned}
 (38) \quad & A_{12}(x_3y_4 - x_4y_3) + A_{13}(x_4y_2 - x_2y_4) \\
 & + A_{14}(x_2y_3 - x_3y_2) + A_{23}(x_1y_4 - x_4y_1) \\
 & + A_{42}(x_1y_3 - x_3y_1) + A_{34}(x_1y_2 - x_2y_1) = 0.
 \end{aligned}$$

Rearranging according to terms in x_i , we obtain

$$\begin{aligned}
 (39) \quad & (+A_{23}y_4 + A_{42}y_3 + A_{34}y_2)x_1 \\
 & + (-A_{13}y_4 + A_{14}y_3 - A_{34}y_1)x_2 \\
 & + (+A_{12}y_4 - A_{14}y_2 - A_{42}y_1)x_3 \\
 & + (-A_{12}y_3 + A_{13}y_2 - A_{23}y_1)x_4 = 0.
 \end{aligned}$$

If the quantities y_i are fixed they determine a point

$$(40) \quad \sum b_i y_i = 0.$$

Then, if we let the quantities x_i vary, equation (39) represents a sphere, the locus of the points of the circles of the complex, which go through the point (40), in other words, the locus of the circles themselves. The point (40) is on the sphere (39), since

$$(41) \quad x_i = y_i$$

satisfies (39). Therefore

To each point of space corresponds, with respect to each complex, a sphere passing through it, and all the circles of the complex passing through the point lie on the sphere.

C. Polar Spheres and Pole Point-Pairs

21. We denote a point and a sphere related as in the last two sections as *pole* and *polar* with respect to the complex.

CORRESPONDENCE II. *Every complex correlates the point-pairs and spheres of the space, so that, with respect to the complex, to every point-pair there corresponds a sphere polar to the point-pair, which contains all the circles of the complex which pass through the point-pair, and to every sphere corresponds a point-pair, pole of the sphere, through which pass all the circles of the complex lying upon the sphere.*

CORRESPONDENCE III. *A triple correspondence is set up by each complex between the point-pairs and the spheres of the space and the circle pencils of the complex, such that a point-pair, a sphere and a circle pencil correspond uniquely and are united in position.*

The pole of the sphere

$$(42) \quad \sum b_i x_i = 0$$

is, from (34), the point whose coördinates are

$$(43) \quad \begin{cases} y_1 = A_{12}b_2 + A_{13}b_3 + A_{14}b_4, \\ y_2 = -A_{12}b_1 + A_{23}b_3 - A_{42}b_4, \\ y_3 = -A_{13}b_1 - A_{23}b_2 + A_{34}b_4, \\ y_4 = -A_{14}b_1 + A_{42}b_2 - A_{34}b_3. \end{cases}$$

The polar of the point y_i is the sphere whose coördinates are

$$(44) \quad \begin{cases} b_1 = A_{23}y_4 + A_{42}y_3 + A_{34}y_2, \\ b_2 = -A_{13}y_4 + A_{14}y_3 - A_{34}y_1, \\ b_3 = A_{12}y_4 - A_{14}y_2 - A_{42}y_1, \\ b_4 = -A_{12}y_3 + A_{13}y_2 - A_{23}y_1. \end{cases} \quad [\text{See (39).}]$$

Let

$$(45) \quad \omega(A) \equiv A_{12}A_{34} + A_{13}A_{42} + A_{14}A_{23}.$$

If the last three of equations (43) be multiplied respectively by A_{34} , A_{42} and A_{23} , and the results be added, we find

$$-\omega(A)b_1 = A_{34}y_2 + A_{42}y_3 + A_{23}y_4.$$

Similarly we obtain

$$-\omega(A)b_2 = -A_{34}y_1 + A_{14}y_3 - A_{13}y_4,$$

$$-\omega(A)b_3 = -A_{42}y_1 - A_{14}y_2 + A_{12}y_4,$$

$$-\omega(A)b_4 = -A_{23}y_1 + A_{13}y_2 - A_{12}y_3.$$

Since only the ratios of the quantities b_i are of concern, these equations are equivalent to (44). Similarly (43) may be gotten from (44).

D. Second Proof of Fundamental Correlation

22. The notion of pole and polar with respect to a complex, is so important that we give another proof of this fundamental correlation.

The equation of a linear complex is*

$$(46) \quad \sum_{i,k} A_{ik} p_{ik} = 0 \quad (i, k = 1, \dots, 4, i \neq k);$$

those of two spheres are

$$(47) \quad \sum a_i x_i = 0, \quad \sum b_i x_i = 0.$$

These two spheres, S and S' , intersect in a circle, whose coördinates are

$$p_{ik} = a_i b_k - a_k b_i.$$

The condition that this circle belong to the complex is that its coördinates satisfy equation (46), i. e.,

$$(48) \quad \sum_{i,k} A_{ik} (a_i b_k - a_k b_i) = 0,$$

or

$$(49) \quad \sum_i (b_i \sum_k A_{ik} a_k) = 0,$$

provided the summation is made under the restrictions

$$(50) \quad A_{ik} = -A_{ki}; \quad A_{ii} = 0.$$

Consider another sphere S'' ,

$$(51) \quad \sum_i l_i x_i = 0,$$

where

* Cf. KOENIGS, *Contributions, etc.*, Annales ... Toulouse, 1888, p. F. 12.

$$(52) \quad l_i = \sum_k A_{ik} a_k.$$

The equation [see (49)]

$$(53) \quad \sum_i (b_i \sum_k A_{ik} a_k) \equiv \sum_i b_i l_i = 0,$$

expresses that the spheres S' and S'' are orthogonal. Furthermore, from (48) and (49),

$$(54) \quad \sum_i (a_i \sum_k A_{ik} a_k) \equiv \sum_i l_i a_i = 0,$$

i. e., the spheres S and S'' are orthogonal. The circle of intersection of S' and S , is therefore orthogonal to S'' ; but the equation of S''

$$\sum_i (\sum_k A_{ik} a_k) x_i = 0,$$

is independent of b_i , and in general different from S_5 . Hence the theorem: *All the circles of a linear complex which lie upon a sphere S are orthogonal to a second sphere S'' , which is called the conjugate of S with respect to the complex.*

Hence the circles of the complex lying upon S are orthogonal to two spheres S'' and S_5 . These circles therefore lie in planes, which pass through the centers of both spheres (Theorem VI). These planes form an axial pencil, whose axis (i. e., the line joining the centers of S'' and S_5), cuts S in two points. The circles of the complex lying upon S all cut the axis twice and therefore must pass through these two points. The theorem follows:

All the circles of the complex which lie on a given sphere pass through a point-pair, the pole of the sphere.

The pole of a sphere

$$(55) \quad \sum_i a_i x_i = 0,$$

may be found as follows: Construct the sphere S''

$$(56) \quad \sum_i (\sum_k A_{ik} a_k) x_i = 0,$$

and join its center to the center of S_5 . The intersections of this line with (55) are the point-pair, pole of (55).

23. It is seen that, in general, but one of the circles of a pencil of circles lying upon a sphere will pass through the pole of the sphere, and belong to the complex. We now prove* that if the circles of a complex, whose equation is

$$(57) \quad F(p_{ik}) = 0,$$

* Cf. PLÜCKER, *Neue Geometrie*, p. 18.

are so distributed that but one in general of the circles of a pencil belongs to the complex, then F is a linear function.

Let

$$\lambda_1 p'_{ik} + \lambda_2 p''_{ik} \quad (\lambda_1, \lambda_2 \text{ arbitrary}),$$

be the coördinates of the pencil. The condition that but one of the circles of the pencil shall belong to F is that the equation

$$F(\lambda_1 p'_{ik} + \lambda_2 p''_{ik}) = 0 \quad (\text{for some } \lambda_1 : \lambda_2)$$

shall be of first degree in $\lambda_1 : \lambda_2$. F is therefore of the form $\Sigma A_{ik} p_{ik} = 0$.

E. Comparison between Circle Geometry and Associate Theories

24. Analytically the two theories are seen to be identical. Transition from one theory to another is effected by the following exchange of notions:

ASSEMBLAGE OF CIRCLES	ASSEMBLAGE OF LINES
Circle	Line
Sphere	Plane
Point-pair	Point

Each system is a four-dimensional assemblage. A circle is determined by two spheres or two point-pairs, just as a line is determined by two planes or two points. The analytical condition that two circles or two lines intersect is the same. The polar theory of sphere and point-pair is precisely that of plane and point. The ∞ circles of a complex lying on a sphere constitute a pencil, as do the ∞ lines of a complex lying in a plane. The relationship can be shown more clearly however, and an actual correspondence be set up between the lines of space and the circles orthogonal to a sphere by means of the double-point geometry of COSSERAT.* He takes as element a pair of points upon the surface of a sphere, and as configuration of reference four circles upon the sphere. This double-point has six coördinates p_{ik} , and if the four circles of reference be taken as the intersection of our four spheres of reference, S_i , with S_5 , the coördinates p_{ik} of the double-point of COSSERAT are precisely the coördinates of the orthogonal circle of S_5 fixed by the double-point (see Theorem XIII). Furthermore, COSSERAT shows that if the four centers of the spheres S_i be taken as the vertices of a tetrahedron of reference the coördinates p_{ik} of the line through the double-point are, save for a factor of proportionality, the coördinates p_{ik} of the double-point. Hence the coördinates of a line and of an orthogonal circle through the same double-point differ only by a factor of proportionality.

* COSSERAT, *Sur le cercle comme un élément générateur de l'espace*, Annales... Toulouse, 1889, E 40.

CORRESPONDENCE IV. *To every sphere orthogonal to S_5 corresponds uniquely the circle of intersection in double-point geometry, and the plane of that circle in line geometry. To every circle orthogonal to S_5 corresponds uniquely a double-point upon S_5 , and the line through the double-point.*

Furthermore with respect to any complex the pole of any plane is the point where it is cut by the line joining the elements of the point-pair, pole of the corresponding sphere with respect to the circle complex whose equation is the same; and reciprocally.

Care must be taken not to confuse point-pair and double-point. The former is two points, one within and one without the sphere S_5 , and uniquely related by the formulæ of inversion with respect to S_5 ; the double-point is a pair of points upon the surface of S_5 , either of which may be again paired with any of the ∞^2 points of the surface. There are ∞^4 double-points and but ∞^3 point-pairs.

25. It is of interest to note that the plane * geometry of the point in space of four dimensions belongs to the same family of reciprocal, four-dimensional geometries, as does both line geometry and the present theory of orthogonal circles. Transition may be effected by the following exchange of notions:

ASSEMBLAGE OF CIRCLES	ASSEMBLAGE OF PLANES
Circle	Plane
Sphere	Line
Point-pair	Lineoid

We return to the development of circle geometry, bidding the reader keep in mind the associate theories to which we have called attention.

CHAPTER IV

CONJUGATE CIRCLES, POLE POINT-PAIRS AND POLAR SPHERES

A. General Theorems

26. *If a point and a sphere are united in position, their respective polar and pole are united in position.*†

For let x'_i and a'_i be respectively a point and a sphere, such that

$$(58) \quad \sum a''_i x'_i = 0.$$

* KEYSER, *Plane Geometry*, etc., op. cit.

† Cf. KOENIGS, *La Géométrie Réglée*, chap. 2. MÖBIUS, *Über Figuren im Raume*, Crelle's Journal, vol. 10, p. 321.

Let a'_i and x''_i be the polar and pole respectively, of x'_i and a''_i with respect to a complex C . To prove

$$\sum a'_i x''_i = 0.$$

The circle of intersection of the spheres a'_i and a''_i , being contained in a'_i and containing x'_i (58), the pole of a'_i , belongs to the complex. Therefore since it lies upon a''_i , it contains x''_i . Therefore

$$\sum a'_i x''_i = 0.$$

27. Let p_{ik} be any circle. Every generating sphere S' of p_{ik} is united in position with every generating point P of p_{ik} . Hence every pole P' of the spheres S' , is united in position with every polar S of the points P . That is, the points, P' , and the spheres, S , generate one and the same circle p'_{ik} . Two circles, p_{ik} and p'_{ik} , so related that each of them is the locus (or envelope) of the poles (or polars) of the generating spheres (or points) of the other, are called *conjugate circles*.

28. The following theorems * result at once by analogy with the line geometry of ordinary space, and the plane geometry of the point in four-space.

Every circle of a complex is self-conjugate with respect to that complex.

Every self-conjugate circle belongs to the complex.

Two distinct conjugate circles cannot pass through a same point, else they would belong to the complex and be self-conjugate.

Two distinct conjugate circles cannot lie on a same sphere.

If two conjugate circles have each a sphere (point) in common with a third circle, this third circle belongs to the complex.

If a circle of the complex $\left\{ \begin{smallmatrix} \text{lies on} \\ \text{passes through} \end{smallmatrix} \right\}$ the same $\left\{ \begin{smallmatrix} \text{sphere} \\ \text{point-pair} \end{smallmatrix} \right\}$ with one of two conjugate circles, it $\left\{ \begin{smallmatrix} \text{lies on} \\ \text{passes through} \end{smallmatrix} \right\}$ a same $\left\{ \begin{smallmatrix} \text{sphere} \\ \text{point-pair} \end{smallmatrix} \right\}$ with the other also.

All the circles intersecting two conjugate circles belong to the complex.

29. *If two circles intersect, their conjugates also intersect.*

For let p' and p'' be two circles intersecting in the point P . The conjugates of p' and p'' lie on a sphere S , polar of P , and therefore intersect.

CORRESPONDENCE V. *To the ∞^2 circles of a point-pair correspond the ∞^2 circles of a sphere polar to that point-pair with respect to a complex, and reciprocally.*

B. Special Complexes of Circles

A complex is called *special* when all its circles intersect a same circle, the *directrix* of the complex. The condition that two circles p_{ik} and p'_{ik} may intersect is

$$(59) \quad p_{12}p'_{34} + p_{13}p'_{42} + \cdots + p_{34}p'_{12} = 0.$$

[See (22).]

* Cf. KOENIGS, *La Géométrie Régliée*, chap. 1. KEYSER, op. cit., p. 417. MÖBIUS, op. cit., etc.

If the quantities p'_{ik} are fixed this is the equation of a complex, all of whose circles intersect p'_{ik} , i. e., a special complex whose directrix is p'_{ik} . The conditions that the complex

$$\sum A_{ik} p_{ik} = 0$$

may be special is therefore

$$(60) \quad \omega(A_{ik}) = 0.$$

32. Every pair of complexes,* C and C' , of which one C' is special, determines another special complex C'' such that the assemblage of circles common to C and C'' , is identical with that common to C and C' .

The director circles of C' and C'' are conjugates with respect to C . C is one of the pencil of complexes determined by

$$\lambda_1 C' + \lambda_2 C'' \quad (\lambda_1, \lambda_2 \text{ arbitrary}).$$

From this may be derived the coördinates of a circle p''_{ik} conjugate to p'_{ik} .

$$(61) \quad \lambda_2 p''_{ik} = \frac{\partial \omega(A)}{\partial A_{ik}} - \frac{\omega(A)}{\sum A_{ik} p'_{ik}} p'_{ik}.$$

Also given a complex C and a circle p_{ik} , the theorem follows:

p_{ik} is self-conjugate or not, according as it belongs, or does not belong to C ; if p'_{ik} and p''_{ik} be any two circles, their conjugates are distinct, or not, according as C is non-special, or special. In case C is special the directrix is conjugate to all circles, itself included.

33. The following theorems† result at once from the definitions of pole and polar.

The polars of the points of a sphere pass through a common point, the pole of the sphere.

The poles of the spheres of a point lie upon a same sphere, the polar of the point.

The conjugates of the circles lying upon a sphere all pass through the pole of the sphere, and the conjugates of the circles passing through a point, all lie upon a sphere, the polar of the point. A circle passing through two points has for its conjugate the circle formed by the intersection of the spheres, polar to these points.

The circle formed by the intersection of two spheres, has for its conjugate the circle passing through the poles of the spheres.

C. Coördinates of Conjugate Circles

34. We derive the coördinates of a circle conjugate to a given circle, by a method slightly different from that used in section 32, and based directly upon the principles of section 33. Given a complex

* Cf. KEYSER, *Plane Geometry, etc.*, pp. 318-320. KOENIGS, *La Géométrie Réglée*, op. cit., p. 24.

† Cf. MÖBIUS, *Über Figuren im Raume*, pp. 329-336.

$$\sum A_{ik} p_{ik} = 0,$$

and a circle p_{ik} defined by the spheres, S and S' ,

$$(62) \quad \begin{cases} p_{12}x_2 + p_{13}x_3 + p_{14}x_4 = 0, \\ p_{21}x_1 + p_{23}x_3 + p_{24}x_4 = 0, \end{cases}$$

we seek to find the coördinates of p'_{ik} conjugate of p_{ik} . The conjugate of p_{ik} is the locus of the poles of the generating spheres of p_{ik} , and is fixed by any two of these poles, particularly by the poles of (62) themselves.

These poles are [see (43)],

$$(63) \quad \begin{cases} x'_1 = A_{12}p_{12} + A_{13}p_{13} + A_{14}p_{14}, \\ x'_2 = A_{23}p_{13} - A_{42}p_{14}, \\ x'_3 = -A_{23}p_{12} + A_{34}p_{14}, \\ x'_4 = A_{42}p_{12} - A_{34}p_{13}; \end{cases}$$

and

$$(64) \quad \begin{cases} x''_1 = A_{13}p_{23} + A_{14}p_{24}, \\ x''_2 = -A_{12}p_{21} + A_{23}p_{23} - A_{42}p_{24}, \\ x''_3 = -A_{13}p_{21} + A_{34}p_{24}, \\ x''_4 = -A_{14}p_{21} - A_{34}p_{23}. \end{cases}$$

A circle passing through these two points is fixed by any two spheres passing through the points, in particular by the spheres passing through these points and one of the points

$$(65) \quad x_2 = x_3 = x_4 = 0, \quad x_1 = x_3 = x_4 = 0,$$

i. e., one of the vertices of the configuration of reference [see (5)]. The equations of these two spheres are

$$(66) \quad \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ x'_1 & x'_2 & x'_3 & x'_4 \\ x''_1 & x''_2 & x''_3 & x''_4 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 0,$$

and

$$(67) \quad \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ x'_1 & x'_2 & x'_3 & x'_4 \\ x''_1 & x''_2 & x''_3 & x''_4 \\ 0 & 1 & 0 & 0 \end{vmatrix} = 0.$$

Expanding we obtain

$$(68) \quad \begin{cases} \alpha_{34}x_2 + \alpha_{42}x_3 + \alpha_{23}x_4 = 0, \\ \alpha_{43}x_1 + \alpha_{14}x_3 + \alpha_{31}x_4 = 0. \end{cases}$$

where

$$\alpha_{ik} = x'_i x''_k - x'_k x''_i.$$

Since the spheres (68) are orthogonal to the fundamental spheres, whose equations are

$$x_1 = 0, \quad x_2 = 0,$$

the quantities α_{ik} may be taken as the coördinates of the circle (see section 9). That is

$$(69) \quad \begin{cases} p'_{12} = \alpha_{34} = x'_3 x''_4 - x'_4 x''_3, \\ p'_{13} = \alpha_{42} = x'_4 x''_2 - x'_2 x''_4, \\ p'_{14} = \alpha_{23} = x'_2 x''_3 - x'_3 x''_2, \\ p'_{23} = \alpha_{14} = x'_1 x''_4 - x'_4 x''_1, \\ p'_{42} = \alpha_{13} = x'_1 x''_3 - x'_3 x''_1. \end{cases}$$

From the form of these equations and the condition

$$\omega(p) = 0,$$

it follows that

$$p'_{34} = \alpha_{12} = x'_1 x''_2 - x'_2 x''_1.$$

But these are the coördinates q_{ik} of the circle regarded as a locus defined by the two points x'_i and x''_i . In fact, the last theorem of section 33 states: *That the conjugate of a circle formed by the intersection of two spheres is the circle fixed by the poles of these spheres.*

The theorem follows:

If a circle is given by coördinates p_{ik} and its conjugate by coördinates q_{ik} , the spheres which define the coördinates p_{ik} are the polars of the points which define the coördinates q_{ik} , and reciprocally.

It is found on trial that the results of (61) and (69) agree save for a factor of proportionality.

D. Construction of the Complex

35. Let the symbols (P, P') and (S, S') denote the circles drawn through the points P and P' , and lying on the spheres S and S' respectively. Let the symbol (P', P'', P''') denote the sphere drawn through the three points P', P'', P''' and (S', S'', S''') denote the point of intersection of the spheres S', S'' and S''' .

36. Five non-intersecting circles of a complex being known, the complex may be determined. Analytically, they give five equations, from which the five constants of the complex may be determined. Geometrically, four of them are cut by but two circles, conjugate with respect to the complex. A different four give another pair of conjugates. Two pairs of conjugates are sufficient to fix the pole of any sphere, or the polar of any point, for every circle cutting a pair of conjugates belongs to the complex, and of the circles cutting a pair of conjugates, one in general passes through any point, or lies on any sphere. The polar or pole is fixed by the two circles of the complex passing through the point or lying on the sphere.

Three spheres and their poles being known, the complex is determined provided not more than one of the circles of intersection belong to the complex i. e., not more than two of the poles lie upon the circles of intersection. For the conjugates of the circles of intersection are the circles determined by the poles of the corresponding pair of spheres. In the exceptional case two of the circles coincide with their conjugates, and the configuration determines but one pair of conjugates.

The general case may be treated as follows: Given three spheres S' , S'' , S''' with poles P' , P'' , P''' , we seek the polar of a point P . Draw the three spheres, (P, P', P'') , (P, P', P''') , (P, P'', P''') . They cut the three circles of intersection, $(S' S'')$, $(S' S''')$, $(S'' S''')$, in the points P_1, P_2, P_3 respectively. These points are the poles of the spheres respectively, for (P', P'') and (S', S'') are conjugate circles, etc. P_1, P_2, P_3 being the poles of three spheres passing through a same point P , all lie on a same sphere (P_1, P_2, P_3) , polar of P , and therefore containing P . The reciprocal problem, given S to find its pole, may be similarly treated. The point (S', S'', S''') is on the sphere (P', P'', P''') , since the poles of all the spheres passing through a point, lie on a sphere which contains that point as pole.

The following table shows these relations:

SPHERE	POLE	SPHERES CONTAINING POLE		
S'	P'	$(P P' P'')$	$(P P' P''')$	$(P' P'' P''')$
S''	P''	$(P P' P'')$	$(P P'' P''')$	$(P' P'' P''')$
S'''	P'''	$(P P' P''')$	$(P P'' P''')$	$(P' P'' P''')$
$(P_1 P_2 P_3)$	P	$(P P' P'')$	$(P P' P''')$	$(P P'' P''')$
$(P P' P'')$	P_1	S'	S''	$(P_1 P_2 P_3)$
$(P P' P''')$	P_2	S'	S''	$(P_1 P_2 P_3)$
$(P P'' P''')$	P_3	S''	S'''	$(P_1 P_2 P_3)$
$(P' P'' P''')$	$(S' S'' S''')$	S'	S''	S'''

The configuration* therefore consists of two groups of four spheres, the poles of each of which are the intersections of three of the spheres of the other group. If such a group be called a *tetrasphere*, the configuration consists of two tetraspheres, each of which is both inscribed in and circumscribed about the other. For the poles of the spheres of each are the vertices of the other.

CHAPTER V

LINEAR CONGRUENCES OF CIRCLES AND PROBLEMS UPON THE ASSEMBLAGE

A. The Linear Congruence

37. A linear congruence consists of the circles whose coördinates satisfy two linear equations of the first degree, i. e., the circles common to two complexes. Let a pair of complexes be

$$(70) \quad \begin{cases} A \equiv \Sigma A_{ik} p_{ik} = 0, \\ B \equiv \Sigma B_{ik} p_{ik} = 0. \end{cases}$$

Upon a sphere there lie in general one pencil of circles satisfying A and another pencil satisfying B . The circle belonging to both of these pencils is in general the only circle on the sphere belonging to the congruence. Reciprocally, through a given point there passes in general one circle of the congruence.

38. The given complexes A and B may be replaced by any two of the pencil of complexes

$$\lambda_1 A + \lambda_2 B = 0 \quad (\lambda_1, \lambda_2 \text{ arbitrary}).$$

Of the complexes of a pencil, two are special, for there are in general two solutions to the quadratic condition

$$(71) \quad \omega(\lambda_1 A + \lambda_2 B) = 0.$$

All the circles of a special complex intersect the directrix (see section 31). Hence the circles of the congruence, being common to the two special complexes, consist precisely of the ∞^2 circles intersecting the two directrices of the special complexes of the pencil. The directrices are called *directrices of the congruence*. The circles passing through any point of one directrix, and intersecting the other, lie on a sphere of which the point is the pole with respect to each of the complexes of the pencil, since the directrices are conjugates with respect to each of the complexes of the pencil. Hence the directrices of a con-

* Cf. MÖBIUS, *Über Figuren im Raume*, Crelle's Journal, Vol. 10, p. 324.

gruence may be defined* as the $\left\{ \begin{smallmatrix} \text{envelope} \\ \text{locus} \end{smallmatrix} \right\}$ of the $\left\{ \begin{smallmatrix} \text{spheres} \\ \text{points} \end{smallmatrix} \right\}$ whose $\left\{ \begin{smallmatrix} \text{poles} \\ \text{polars} \end{smallmatrix} \right\}$ are the same with respect to every complex of the pencil, of which the congruence is the common part. Any two circles conjugate with respect to a complex may be taken as directrices of a congruence belonging to the complex. Again, the directrices may be defined as, the locus of points (envelope of spheres) through which ∞ circles of the congruence pass. The equation of condition

$$(71) \quad \omega(\lambda_1 A + \lambda_2 B) = 0$$

may have two real roots, giving a congruence with real directrices, two imaginary roots, giving imaginary directrices, or two coincident roots, in which case the directrices coincide, the double directrix belongs to the congruence, and if a bilinear relation be set up between the generating points and the generating spheres of the directrix, then the assemblage of circles obtained by taking all and only the pencils that are contained on the spheres corresponding to the points, constitute the congruence.† In case (71) gives an indeterminate solution for $\lambda_1 : \lambda_2$, the congruence has an infinity of directrices, which constitute a pencil of circles. The congruence consists of the circles of the point common to the circles of the pencil, and of the circles of the sphere upon which the pencil lies.

The theory of the congruence of circles is seen to be identical in form with those of the congruence of lines, and of the congruence of planes in point geometry of four space.‡

38. We now turn to certain special problems relating to the complex, and first find the condition that a complex shall contain a given congruence. Let the complex be given by

$$\sum A_{ik} p_{ik} = 0,$$

and the congruence by

$$\sum B_{ik} p_{ik} = 0, \quad \sum C_{ik} p_{ik} = 0.$$

The condition is, that the complex shall be one of the pencil of complexes defined by

$$\sum (\lambda_1 B_{ik} + \lambda_2 C_{ik}) p_{ik} = 0,$$

i. e., for some value of $\lambda_1 : \lambda_2$ the six equations

$$\lambda_1 B_{ik} + \lambda_2 C_{ik} = A_{ik}$$

shall be consistent. This is equivalent to the four independent conditions given by

* Cf. PLÜCKER, *Neue Geometrie*, p. 78.

† Cf. KOENIGS, *La Géométrie Régée*, chap. 3, *Annales*... Toulouse, 1892; PLÜCKER, *Neue Geometrie*, p. 62.

‡ See references to PLÜCKER, KOENIGS, KEYSER, etc.

$$(72) \quad \begin{vmatrix} A_{12} & B_{12} & C_{12} \\ A_{13} & B_{13} & C_{13} \\ A_{14} & B_{14} & C_{14} \\ A_{23} & B_{23} & C_{23} \\ A_{42} & B_{42} & C_{42} \\ A_{34} & B_{34} & C_{34} \end{vmatrix} = 0.$$

A complex may be subjected to five conditions. There are therefore ∞ complexes containing a given congruence i. e., the complexes of the pencil whose directrices define the congruence.

The condition that a complex may contain a given circle p'_{ik} is evidently

$$(73) \quad \sum A_{ik} p'_{ik} = 0.$$

Similarly a complex will contain a given pencil

$$\lambda_1 p'_{ik} + \lambda_2 p''_{ik}$$

if it contains the generating circles of the pencil, i. e.,

$$\sum A_{ik} p'_{ik} = 0, \quad \sum A_{ik} p''_{ik} = 0.$$

The systems defined by the equations

$$\sum B'_{ik} p_{ik} = 0, \dots \sum B^n_{ik} p_{ik} = 0 \quad (n \leq 4),$$

belong to the complex, if

$$(74) \quad \begin{vmatrix} A_{12} & B'_{12} & \dots & B^n_{12} \\ \cdot & \cdot & \cdot & \cdot \\ A_{34} & B'_{34} & \dots & B^n_{34} \end{vmatrix} = 0,$$

giving $(6 - n)$ independent conditions. In the case $n = 4$, the four equations above, and the quadratic identity define two circles, which accounts for the $6 - 4 = 2$ conditions.

B. The Surface with Circles as Generators Formed by the Intersection of Two Congruences

39. Two congruences of a complex of lines intersect in a ruled surface, the hyperboloid of one sheet, having two systems of straight-line generators. In section 24 (p. 23) we set up a one-to-one correspondence between the lines of space and the assemblage of circles orthogonal to a given sphere. The analytical condition that two circles intersect is precisely the condition that the two corresponding lines intersect. Hence by analogy we derive the properties of the surface formed

by the intersection of two congruences of circles of a complex. Consider two congruences of a complex, as defined by two pairs of circles p_1, p'_1 and p_2, p'_2 , conjugate with respect to the complex. Any circle intersecting two conjugate circles, belongs to the complex, and to the congruence of which the two conjugate circles are directrices. Of these ∞^2 circles, ∞^1 circles intersect any other circle, and two of these ∞^2 circles in general intersect any pair of circles. If this latter pair are conjugate, with respect to the complex, all the circles of the complex which intersect one intersect the other also. The theorem follows:

∞^1 circles of a complex intersect two pairs of circles conjugate with respect to the complex, i. e., two congruences of a complex intersect in a configuration composed of ∞^1 circles of the complex.

These circles generate a surface such that through every point on the surface pass two generating circles. The circles are arranged in two systems constituting a net of circles. No two circles of the same system intersect, but every circle of the one system intersects every circle of the other system. A sphere which contains one circle of the surface contains a second circle of the surface, and may be considered as tangent to the surface at the point-pair in which the two generators intersect. Since no two circles of the same system intersect no sphere can contain two generators of the same system.

In general two congruences do not belong to a same complex. The condition is that one of the generating complexes of one congruence, shall be a generating complex of the other congruence. Three complexes, however, intersect in a surface of the above character. Its equations are therefore given as three equations of the form $\sum A_{ik} p_{ik} = 0$.

C. Radius of a Circle, Circles in Involution, Degeneration of S_5

We treat in this division several special problems upon circles and the assemblage.

40. The radius ρ of a sphere

$$\sum a_i x_i = 0$$

is

$$(75) \quad \rho^2 = \frac{\sum a_i^2}{\left(\sum \frac{a_i}{R_i}\right)^2}.*$$

Every sphere $\dagger$ passing through a circle p_{ik} is given by the equation

$$(76) \quad \sum_i (\lambda p_{\alpha i} + \mu p_{\beta i}) x_i = 0,$$

where α, β are two of the indices 1, 2, 3, 4, and λ and μ are arbitrary. The radius of this sphere is

* DARBOUX, *Théorie des Surfaces*, vol. I, p. 227.

$\dagger$ KOENIGS, *Contributions*, etc., *Annales* ... Toulouse, 1888, p. F 6.

$$(77) \quad \rho^2 = \frac{\sum_i (\lambda p_{\alpha i} + \mu p_{\beta i})^2}{\left[\lambda \sum_i \frac{p_{\alpha i}}{R_i} + \mu \sum_i \frac{p_{\beta i}}{R_i} \right]^2}.$$

When this is a minimum the radius of the sphere will be equal to that of the circle. Its value will be

$$(78) \quad \rho^2 = \frac{\sum_{ij} p_{ij}^2}{\sum_j \left(\sum_i \frac{p_{ij}}{R_i} \right)^2}.$$

When $\rho = 0$

$$(79) \quad \sum_{ij} p_{ij}^2 = \Xi(p) = 0.$$

This is a complex of the second order, and consists of the points of S_5 , regarded as circles orthogonal to S_5 . Each of the points is taken infinity times and in fact is a degenerate pencil of circles, having as axis the radius of S_5 through the point.

When $\rho = \infty$,

$$(80) \quad \sum_j \left(\sum_i \frac{p_{ij}}{R_i} \right)^2 = 0.$$

This is therefore the equation of the straight lines passing through the center of S_5 , each taken ∞ times, i. e., regarded as a degenerate pencil of circles orthogonal to S_5 and of infinite radius. If

$$R_1^2 = R_2^2 = \dots = R_4^2,$$

equation (80) degenerates into

$$\sum_i p_{ij} = 0. \quad (i, j = 1, \dots, 4).$$

41. The polar of the form $\Xi(p)$,

$$(81) \quad \Xi(p, p') \equiv \frac{1}{2} \sum_{ik} \frac{\partial \Xi(p)}{\partial p_{ik}} p'_{ik} \equiv \sum p_{ik} p'_{ik}$$

has been shown by Koenigs* to play an important rôle in the theory of circles in space. Let there be given two circles p_{ik} and p'_{ik} . If through one of them a sphere can be drawn orthogonal to the other, then through the second a sphere can be drawn orthogonal to the first, and the circles are said to be *in involution*. The analytical condition is found to be

$$(82) \quad \Xi(p, p') = 0.$$

* KOENIGS, *Contributions, etc.*, Annales... Toulouse, 1888, p. F 9.

The transition from the theory of circles in space, to the theory of circles orthogonal to a given sphere, consists solely in a proper choice of fundamental spheres, and the limiting of the range of i, k to the values 1, 2, 3, 4, instead of 1, 2, 3, 4, 5. If one of the circles, p_{ik} be fixed, the circles in involution with it form a special complex, since they are given by the equation

$$(83) \quad \sum p'_{ik} p_{ik} = 0,$$

[see (82)], and the coördinates of the complex are the coördinates of a circle [see (60)]. The quantities p'_{ik} are not the coördinates of the directrix of the complex, regarded as an envelope of spheres; but writing in full equation (83),

$$p'_{12}p_{12} + p'_{13}p_{13} + p'_{14}p_{14} + p'_{23}p_{23} + p'_{42}p_{42} + p'_{34}p_{34} = 0,$$

and the condition that the circles p'_{ik} and p_{ik} intersect [see (22)],

$$p'_{34}p_{12} + p'_{42}p_{13} + p'_{23}p_{14} + p'_{14}p_{23} + p'_{13}p_{42} + p'_{12}p_{34} = 0,$$

it is readily seen that the coördinates p'_{ik} are proportional to the coördinates q'_{ik} , which give the directrix as the locus of points [see (19)]. Hence the theorem: *The circles in involution with a circle p'_{ik} form a special complex whose directrix is given by coördinates q'_{ik} proportional to the coördinates p'_{ik} .*

The same reasoning applies to the coördinates A_{ik} of any special complex [see (60)]

$$\sum A_{ik} p_{ik} = 0.$$

The theorem follows:

The coördinates A_{ik} of a special complex are proportional to the coördinates q_{ik} of the directrix.

42. The angle between two spheres

$$\sum a_i x_i = 0, \quad \text{and} \quad \sum b_i x_i = 0$$

is given by

$$(84) \quad \nu = \cos^{-1} \frac{\sum a_i b_i}{\sqrt{\sum a_i^2} \sqrt{\sum b_i^2}}.*$$

If the spheres are tangent externally or internally

$$\sum a_i b_i = \pm \sqrt{\sum a_i^2} \sqrt{\sum b_i^2}.$$

Squaring and expanding we obtain

$$(85) \quad \sum (a_i b_k - a_k b_i)^2 = \sum p_{ik}^2 = 0.$$

Conversely, if

$$\sum p_{ik}^2 = 0,$$

the spheres are tangent. Two spheres, however, can only be tangent at a point on the sphere S_5 . Hence (85) is the equation of the points of S_5 , regarded as

* DARBOUX, *Théorie des Surfaces*, vol. I, p. 228.

circles orthogonal to S_5 . Each point is taken ∞ times, since a pair of spheres may be tangent at any point on S_5 in ∞ positions. If the quantities p_{ik} are real, they must all equal zero, and their ratios be indeterminate. These results are identical with that of (79).

43. The system as a whole may degenerate in two ways, — the radius of S_5 may become zero or infinity. In the former case S_5 becomes a point. Each of the ∞^3 spheres and ∞^4 circles passing through the point may be regarded as orthogonal to it. The inside elements of all the point-pairs coincide and this point taken ∞^3 times is regarded as related by inversion to every point of space. The formulæ of inversion of course become indeterminate.

Secondly, if the radius of S_5 becomes infinite, S_5 becomes a plane. The ∞^3 spheres and ∞^4 circles orthogonal to the plane, all have their centers in the plane. Points on opposite sides of the plane, and symmetrically placed, form a point-pair, their axis being perpendicular to the plane, just as the axis of a point-pair in the general case is a radius of the sphere and therefore perpendicular to the surface of the sphere.

CHAPTER VI

FAMILIES OF SPHERES

44. The spheres for which the set of ratios

$$a_1 : a_2 : a_3$$

is constant are said to form a *family of spheres*. We shall show that families of spheres exhibit many of the characteristics of families of parallel planes in line geometry.

The pole of the sphere

$$\sum a_i x_i = 0$$

is, [see (43)],

$$(86) \quad \begin{cases} x_1 = A_{12}a_2 + A_{13}a_3 + A_{14}a_4, \\ x_2 = -A_{12}a_1 + A_{23}a_3 - A_{42}a_4, \\ x_3 = -A_{13}a_1 - A_{23}a_2 + A_{34}a_4, \\ x_4 = -A_{14}a_1 + A_{42}a_2 - A_{34}a_3 \equiv A. \end{cases}$$

The sphere

$$(87) \quad a_1x_1 + a_2x_2 + a_3x_3 = 0$$

belongs to the same family and is orthogonal to the fundamental sphere S_4 .

The pole of (87) is

$$(88) \quad \begin{cases} x'_1 = A_{12}a_2 + A_{13}a_3, \\ x'_2 = -A_{12}a_1 + A_{23}a_3, \\ x'_3 = -A_{13}a_1 - A_{23}a_2, \\ x'_4 = -A_{14}a_1 + A_{42}a_2 - A_{34}a_3 \equiv A. \end{cases}$$

From (86) and (88) we obtain

$$(89) \quad \begin{cases} x_1 - x'_1 = A_{14} a_4, \\ x_2 - x'_2 = -A_{42} a_4, \\ x_3 - x'_3 = A_{34} a_4, \end{cases}$$

Dividing the several terms of these equations by

$$x_4 = x'_4 = A,$$

we obtain

$$(90) \quad \begin{cases} \frac{x_1}{x_4} - \frac{x'_1}{x'_4} = \frac{A_{14} a_4}{A}, \\ \frac{x_2}{x_4} - \frac{x'_2}{x'_4} = -\frac{A_{42} a_4}{A}, \\ \frac{x_3}{x_4} - \frac{x'_3}{x'_4} = \frac{A_{34} a_4}{A}. \end{cases}$$

whence

$$(91) \quad \frac{x_1 x'_4 - x_4 x'_1}{A_{14}} = \frac{x_2 x'_4 - x_4 x'_2}{-A_{42}} = \frac{x_3 x'_4 - x_4 x'_3}{A_{34}}.$$

If we regard x_i as variable, it is seen that these are the equations of a circle, the locus of points x_i the poles of the spheres of the family

$$a_1 : a_2 : a_3.$$

For since the equations are independent of a_4 , they are satisfied by the coördinates x_i of the pole of any sphere of the form

$$(92) \quad a_1 x_1 + a_2 x_2 + a_3 x_3 + \lambda x_4 = 0 \quad (\lambda \text{ arbitrary}).$$

Similarly the poles of the spheres of the family

$$(93) \quad b_1 x_1 + b_2 x_2 + b_3 x_3 + \lambda x_4 = 0$$

lie on a circle whose equations are

$$(94) \quad \frac{x_1 y'_4 - x_4 y'_1}{A_{14}} = \frac{x_2 y'_4 - x_4 y'_2}{-A_{42}} = \frac{x_3 y'_4 - x_4 y'_3}{A_{34}},$$

where the point y'_i is the pole of the sphere

$$b_1 x_1 + b_2 x_2 + b_3 x_3 = 0.$$

We shall now show that the two circles (91) and (94) lie on a same sphere. Expanding we have as the equations of the spheres, S and S' , defining (94),

$$(95) \quad \begin{cases} -A_{42} y'_4 x_1 - A_{14} y'_4 x_2 + (A_{42} y'_1 + A_{14} y'_2) x_4 = 0, \\ A_{34} y'_4 x_1 - A_{14} y'_4 x_3 + (A_{14} y'_3 - A_{34} y'_1) x_4 = 0. \end{cases}$$

Similarly from (91) we obtain,

$$(96) \quad \begin{cases} -A_{42}x'_4x_1 - A_{14}x'_4x_2 + (A_{42}x'_1 + A_{14}x'_2)x_4 = 0, \\ A_{34}x'_4x_1 - A_{14}x'_4x_3 + (A_{14}x'_3 - A_{34}x'_1)x_4 = 0. \end{cases}$$

In order that the circles may lie on a same sphere, it is necessary and sufficient that one of the pencil of spheres

$$S + \lambda S' \quad (\lambda \text{ arbitrary}),$$

defined by (95), should be identical with one of the pencil

$$S'' + \mu S''' \quad (\mu \text{ arbitrary}),$$

defined by (96), for some values of λ and μ . That is, equating ratios of coefficients

$$(97) \quad \left\{ \begin{aligned} & \frac{-A_{42}y'_4 + \lambda A_{34}y'_4}{y'_4 A_{14}} = \frac{-A_{42}x'_4 + \mu A_{34}x'_4}{x'_4 A_{14}}, \\ & -\lambda = -\mu, \\ & \frac{(A_{42}y'_1 + A_{14}y'_2) + \lambda(A_{14}y'_3 - A_{34}y'_1)}{y'_4 A_{14}} \\ & \quad = \frac{(A_{42}x'_1 + A_{14}x'_2) + \mu(A_{14}x'_3 - A_{34}x'_1)}{x'_4 A_{14}}. \end{aligned} \right.$$

The second of these equations of condition requires that

$$\lambda = \mu.$$

Substituting this value in the first equation, it is seen to be satisfied identically whatever the value of μ . There remains but one equation and μ may be taken so as to satisfy it. The theorem follows:

The loci of the poles of two families of spheres are two circles lying upon a same sphere.

We call such circles *diametral circles*.

45. From the last of equations (97) we obtain

$$\mu = \frac{y'_4(x'_1 A_{42} + x'_2 A_{14}) - x'_4(y'_1 A_{42} + y'_2 A_{14})}{x'_4(y'_3 A_{14} - y'_1 A_{34}) - y'_4(x'_3 A_{14} - x'_1 A_{34})},$$

which reduces to

$$\begin{aligned} \mu &= \frac{A_{14}\omega(A)p_{13} - A_{42}\omega(A)p_{23}}{A_{14}\omega(A)p_{12} - A_{34}\omega(A)p_{23}}, \\ &= \frac{A_{14}p_{13} - A_{42}p_{23}}{A_{14}p_{12} - A_{34}p_{23}}. \end{aligned}$$

When the complex is special, μ is indeterminate. Otherwise μ is a function of A_{14} , A_{42} and A_{34} and of a_1 , a_2 , a_3 and b_1 , b_2 , b_3 , and changes as these latter quantities change, i. e., is different for different families. It follows that *the*

circles associated with different families do not all lie on a same sphere, but have a relation similar to that of parallel straight lines in space. Any two of these lines lie in a plane just as any two of the circles lie in a sphere, but in general three lines or three circles do not lie in a same plane or on a same sphere. PLÜCKER has shown that the locus of the poles of a system of parallel planes is a straight line ("Durchmesser"), and all these diameters are parallel. A family of spheres in the circle theory is seen to correspond to a system of parallel planes in the line theory.*

46. *There are ∞^2 families, since*

$$a_1 : a_2 : a_3$$

involves two parameters. There are ∞^2 diametral circles, since in general each family defines its diametral circle uniquely.

Upon each of the generating spheres of a diametral circle, lie ∞ diametral circles.

There are ∞^2 circles, not lying on the common sphere of two given diametral circles, but co-spherical with each of them separately. These circles are co-spherical with every diametral circle.

For consider a circle given by a sphere of the pencil (95)

$$(98) \quad S + \lambda S',$$

and by a sphere of the pencil defined by (96)

$$(99) \quad S'' + \mu S'''.$$

Let the equations of any other diametral circle be

$$(100) \quad \begin{cases} -A_{42}z'_4x_1 - A_{14}z'_4x_2 + (A_{42}z'_1 + A_{14}z'_2)x_4 = 0, \\ A_{34}z'_4x_1 - A_{14}z'_4x_3 + (A_{14}z'_3 - A_{34}z'_1)x_4 = 0. \end{cases}$$

The condition that the circle defined by spheres (98) and (99) shall intersect the diametral circle (100) is the vanishing of the determinant of the coefficient of these four spheres i. e., the condition that they have a point in common,

$$(101) \quad \begin{vmatrix} A_{34}x'_4 - \rho A_{42}x'_4 & -\rho A_{14}x'_4 & -A_{14}x'_4 & \rho A_{42}x'_1 \text{ etc.} \\ A_{34}y'_4 - \rho A_{42}y'_4 & -\rho A_{14}y'_4 & -A_{14}y'_4 & \rho A_{42}y'_1 \text{ etc.} \\ -A_{42}z'_4 & -A_{14}z'_4 & 0 & A_{42}z'_1 \text{ etc.} \\ A_{34}z'_4 & 0 & -A_{14}z'_4 & A_{14}z'_3 \text{ etc.} \end{vmatrix}.$$

Taking the factor $-A_{14}$ from the second and third columns and replacing it by $-A_{42}$ and A_{34} , we obtain

* PLÜCKER, *Neue Geometrie*, etc., pp. 35 et al. Also MÖBIUS, *Crelle's Journal*, vol. 10, pp. 329 et al.

$$-\frac{A_{14}^2}{A_{42}A_{34}} \begin{vmatrix} A_{34}x'_4 \text{ etc.} & -\rho A_{42}x'_4 & A_{34}x'_4 & \rho A_{42}x'_1 \text{ etc.} \\ A_{34}y'_4 \text{ etc.} & -\rho A_{42}x'_4 & A_{34}y'_4 & . \\ -A_{42}z'_4 & -A_{42}z'_4 & 0 & . \\ A_{34}z'_4 & 0 & A_{34}z'_4 & . \end{vmatrix}.$$

Adding third column to second, we obtain

$$-\frac{A_{14}^2}{A_{42}A_{34}} \begin{vmatrix} A_{34}x'_4 - \rho A_{42}x'_4 & A_{34}x'_4 - \rho A_{42}x'_4 & A_{34}x'_4 & . \\ A_{34}y'_4 - \rho A_{42}y'_4 & A_{34}x'_4 - \rho A_{42}z'_4 & A_{34}y'_4 & . \\ -A_{42}z'_4 & -A_{42}z'_4 & 0 & . \\ A_{34}z'_4 & A_{34}z'_4 & A_{34}z'_4 & . \end{vmatrix} \equiv 0.$$

The theorem follows:

A circle co-spherical with two diametral circles, but not lying on their common sphere, is co-spherical with every diametral circle.

In line geometry every line co-planar with two diameters but not in their common plane, i. e., parallel to them, is a diameter. Hence by analogy each of these ∞^2 circles is a diametral circle.

CHAPTER VII

TRANSFORMATIONS OF THE ASSEMBLAGE

A. Generalization of Coördinates γ_{ik}

47. "The coördinates γ_{ik} " of section 11, "admit of generalization.* We know from the theory of forms, that the new variables ν_i in the transformation,

$$\rho\gamma_{ik} = C_{ik,1}\nu_1 + C_{ik,2}\nu_2 + \cdots C_{ik,6}\nu_6,$$

where the modulus is not zero, are connected by a homogeneous quadratic identity

$$\Omega(\nu) \equiv 0,$$

where $\Omega(\nu)$ is the transformed of $\omega(\gamma)$. Moreover the polar form $\omega(\gamma, \gamma')$ of $\omega(\gamma)$ is converted by the same transformation into the polar form,

$$\Omega(\nu, \nu'),$$

of $\Omega(\nu)$. It is well known that $\omega(\gamma)$, regarded as a quadratic form, has a non-vanishing discriminant, and that it is possible to find a linear transformation which will convert this form into any quadratic form $\Omega(\nu)$, whose discrim-

* See KEYSER, *Plane Geometry, etc.*, p. 310. KOENIGS, *La Géométrie Régliée*.

inant does not vanish. Accordingly we may employ for homogeneous" circle "coördinates any six variables ν_i connected by the quadratic identity $\Omega(\nu)$, where $\Omega(\nu)$ has a non-vanishing discriminant. Hence the condition that the" circles " ν_i and ν'_i shall intersect in a" point pair "or lie on a same" sphere "is that

$$\Omega(\nu, \nu') = 0."$$

B. Transformation of Fundamental Spheres

48. We now consider the general transformation of the fundamental spheres S_i . The equation of a sphere in space in pentaspherical coördinates is

$$(102) \quad S \equiv a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 + a_5 x_5 = 0,$$

where

$$(103) \quad \rho x_i = \frac{P_i}{R_i},$$

P_i being the power of the point x_i with respect to the sphere S_i , and R_i being the radius of S_i . Consider the linear transformation

$$(104) \quad \begin{cases} a_1 = \alpha_{11} a'_1 + \alpha_{12} a'_2 + \alpha_{13} a'_3 + \alpha_{14} a'_4 + \alpha_{15} a'_5, \\ a_2 = \alpha_{21} a'_1 + \alpha_{22} a'_2 + \alpha_{23} a'_3 + \alpha_{24} a'_4 + \alpha_{25} a'_5, \\ a_3 = \alpha_{31} a'_1 + \alpha_{32} a'_2 + \alpha_{33} a'_3 + \alpha_{34} a'_4 + \alpha_{35} a'_5, \\ a_4 = \alpha_{41} a'_1 + \alpha_{42} a'_2 + \alpha_{43} a'_3 + \alpha_{44} a'_4 + \alpha_{45} a'_5, \\ a_5 = \alpha_{51} a'_1 + \alpha_{52} a'_2 + \alpha_{53} a'_3 + \alpha_{54} a'_4 + \alpha_{55} a'_5, \end{cases} \quad (\alpha_{55} = 1),$$

which involves twenty-four parameters. The sphere is transformed into

$$(105) \quad S' \equiv \sum_{1i}^5 x_i \sum_{1k}^5 \alpha_{ik} a'_k \equiv \sum_{1k}^5 a'_k \sum_{1i}^5 \alpha_{ik} x_i \equiv \sum_{1k}^5 a'_k x'_k = 0$$

where

$$x'_k = \sum_{1i}^5 \alpha_{ik} x_i.$$

We now seek the conditions that a point $P(xyz)$ represented by x_i , shall be represented by x'_k . That is

$$(106) \quad \rho x'_k = \frac{P'_k}{R'_k} \quad (k=1, \dots, 5),$$

where P'_k is the power of the point $P(xyz)$ with respect to a new fundamental sphere S'_k , and R'_k is the radius of that sphere. Expanding x_i we obtain

$$(107) \quad x_i \equiv \frac{x^2 + y^2 + z^2 + 2A_i x + 2B_i y + 2C_i z + A_i^2 + B_i^2 + C_i^2 - R_i^2}{R_i},$$

whence

$$(108) \quad \begin{aligned} x'_k \equiv \sum_i \alpha_{ik} x_i &= \sum_i \frac{\alpha_{ik}}{R_i} (x^2 + y^2 + z^2) \\ &+ 2 \sum_i \frac{A_i \alpha_{ik}}{R_i} x + \cdots \sum_i \frac{\alpha_{ik}}{R_i} (A_i^2 + B_i^2 + C_i^2 - R_i^2). \end{aligned}$$

Write $\alpha_{ik}/R_i \equiv \beta_{ik}$, and divide numerator and denominator by $\sum_i \beta_{ik}$,

$$(109) \quad x'_k = \frac{x^2 + y^2 + z^2 + \frac{2 \sum_i A_i \beta_{ik} x + \cdots + \sum_i \beta_{ik} (A_i^2 + B_i^2 + C_i^2) - \sum_i \beta_{ik} R_i^2}{\sum_i \beta_{ik}}}{\frac{1}{\sum_i \beta_{ik}}}.$$

Adding and subtracting

$$(110) \quad \frac{(\sum_i A_i \beta_{ik})^2 + \cdots + (\sum_i C_i \beta_{ik})^2}{(\sum_i \beta_{ik})^2}$$

in the numerator, it becomes of the form

$$(111) \quad x'_k = \frac{(x+a)^2 + (y+b)^2 + (z+c)^2 - r^2}{n},$$

where if x'_k is of the form (106)

$$\rho n^2 = r^2 \quad (\rho \text{ arbitrary}),$$

i. e.,

$$(112) \quad \frac{\rho}{(\sum_i \beta_{ik})^2} = \frac{(\sum_i A_i \beta_{ik})^2 + \cdots + \sum_i \beta_{ik} \sum_i \beta_{ik} R_i^2 - \sum_i \beta_{ik} \sum_i \beta_{ik} (A_i^2 + \cdots + C_i^2)}{(\sum_i \beta_{ik})^2}.$$

This imposes one condition upon the quantities β_{ik} , i. e., on α_{ik} . Hence in general that the five quantities x'_k may represent the same point $P(xyz)$ that x_i represents, it is necessary and sufficient that the quantities α_{ik} be subjected to four conditions of the form (112), and the transformation is equivalent to a change of fundamental spheres.

We next impose the condition that the sphere S_5 may remain unchanged, i. e., that the system of orthogonal circles which compose the space may remain unchanged. It is necessary and sufficient in order that

$$x_5 \equiv x'_5,$$

that

$$(113) \quad \alpha_{15} = \alpha_{25} = \alpha_{35} = \alpha_{45} = 0 \quad \text{and} \quad \alpha_{55} = 1.$$

Impose further the ten conditions that the five spheres, S'_i , shall be mutually orthogonal. Since the four fundamental spheres

$$S'_1, S'_2, S'_3, S'_4$$

are orthogonal to $S'_5 \equiv S_5$, it follows that every sphere of the form

$$(114) \quad \sum_1^4 a'_k x'_k = 0$$

is orthogonal to S_5 . The theorem follows:

The system of circles orthogonal to a given sphere may be transformed into themselves referred to form new fundamental spheres of the system, by a general linear transformation of the quantities a_i , the twenty-four parameters of the transformation being subject to eighteen conditions.

49. As a special case let the sphere S_5 be unchanged, and the quantities a_i ($i = 1, 2, 3, 4$) be transformed by the relations

$$(115) \quad \begin{cases} a_1 = \alpha_{11} a'_1 + \cdots \alpha_{14} a'_4, \\ a_2 = \alpha_{21} a'_1 + \cdots \alpha_{24} a'_4, \\ a_3 = \alpha_{31} a'_1 + \cdots \alpha_{34} a'_4, \\ a_4 = \alpha_{41} a'_1 + \cdots \alpha_{44} a'_4. \end{cases} \quad (\alpha_{44} = 1).$$

This involves fifteen parameters. S becomes

$$(116) \quad S' \equiv \sum_{1i}^4 x_i \sum_{1k}^4 \alpha_{ik} a'_k \equiv \sum_{1k}^4 a'_k \sum_{1i}^4 \alpha_{ik} x_i \equiv \sum_k a'_k x'_k = 0.$$

If x'_k and x_i represent the same point $P(xyz)$, the quantities, α_{ik} , are subjected to three conditions. If the new fundamental spheres S'_i , ($i = 1, \dots, 4$), are orthogonal, six more conditions are imposed. As before there are left six degrees of freedom.

C. Invariance of $\omega(A)$

50. We now show the invariance of the fundamental form, $\omega(A)$, of the complex. Consider the complex

$$(117) \quad \Sigma A_{ik} p_{ik} = 0.$$

A linear substitution

$$(118) \quad a_i = \alpha_{i1} a'_1 + \alpha_{i2} a'_2 + \alpha_{i3} a'_3 + \alpha_{i4} a'_4 \quad (i = 1, \dots, 4)$$

has been shown to be equivalent to a change of fundamental spheres. Let

$$(119) \quad a_i = \sum_k \alpha_{ik} a'_k, \quad b_i = \sum_k \alpha_{ik} b'_k.$$

By this substitution

$$p_{ik} = a_i b_k - a_k b_i$$

becomes

$$(120) \quad p_{ik} = \sum_{jr} l_{ikjr} p'_{jr}$$

where

$$(121) \quad p'_{jr} = a'_j b'_r - a'_r b'_j, \quad l_{ikjr} = \alpha_{ij} \beta_{kr} - \alpha_{kr} \beta_{ij}.$$

Substituting these values in the equation of the complex, we obtain

$$(122) \quad \Sigma A'_{jr} p'_{jr} = 0,$$

where

$$(123) \quad A'_{jr} = \sum_{ik} l_{ikjr} A_{ik}.$$

From these form the invariant

$$(124) \quad \begin{aligned} \omega(A') &= \sum_{ik} \omega_{jr}(l_{ikjr}) A_{ik}^2 + \sum_{ik} \omega_{jr}(l_{ikjr}, l_{ik'jr}) A_{ik} A_{ik'} + \omega_{jr}(l_{12jr}, l_{34jr}) A_{12} A_{34} \\ &\quad + \omega_{jr}(l_{13jr}, l_{42jr}) A_{13} A_{42} + \omega_{jr}(l_{14jr}, l_{23jr}) A_{14} A_{23} \\ &\equiv \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{vmatrix} \omega(A) \equiv \Delta \omega(A). \end{aligned}$$

For terms of the first two types vanish identically, and the coefficients of the last three terms are all equal to Δ . Since $\omega(A)$ is in general not zero, only four of the coefficients of a non-special complex can become zero through any transformation, and the two remaining must not have a common subscript. The complex may take any of the three forms

$$(125) \quad \begin{cases} A'_{12} p_{12} + A'_{34} p_{34} = 0, \\ A'_{13} p_{13} + A'_{42} p_{42} = 0, \\ A'_{14} p_{14} + A'_{23} p_{23} = 0. \end{cases}$$

There are still two degrees of freedom among the coördinates of the transformation. It will be remembered, however, that the reduction of the complex of lines to its simplest equation, is of such a character that the complex may undergo symmetrical translations and rotations with respect to its axis, the equation remaining in its simplest form. Hence in both the cases of line and circle complexes there remain two degrees of freedom.

51. Since the elements of a point-pair are simply interchanged by the formulæ of inversion, it follows that every point-pair, sphere or circle of the assemblage is invariant under this transformation. The theorem follows:

Every configuration of the assemblage is invariant under the transformation by inversion with respect to the fundamental sphere S_5 .

D. Projective Transformation by Means of Complexes

52. We have* seen that any linear complex C sets up a one-to-one correspondence between the spheres and the point-pairs of the assemblage. Let p and

* Compare KEYSER, *Plane Geometry*, etc.

p' be two circles conjugate with respect to C . Each is the $\{\text{envelope}\}_{\{\text{locus}\}}$ of the $\{\text{polars}\}_{\{\text{poles}\}}$ of the $\{\text{point-pairs}\}_{\{\text{spheres}\}}$ of the other. Consequently if S_1, S_2, S_3, S_4 be four spheres of p , their poles P_1, P_2, P_3, P_4 lie upon p' and

$$(S_1 S_2 S_3 S_4) = (P_1 P_2 P_3 P_4),$$

i. e., every complex sets up a projective transformation between the spheres and point-pairs of each pair of circles conjugate with respect to it, and between the spheres and point-pairs of each self-conjugate circle, i. e., each circle belonging to the complex.

If p be a circle common to two complexes, and $S_1, \dots, S_4$ be any four of its spheres whose poles with respect to one of the complexes are $P_1 \dots P_4$, and with respect to the other $P'_1, \dots, P'_4$, then

$$(P_1 P_2 P_3 P_4) = [(S_1 S_2 S_3 S_4)] = (P'_1 P'_2 P'_3 P'_4).$$

Similarly

$$(S_1 S_2 S_3 S_4) = [(P_1 P_2 P_3 P_4)] = (S'_1 S'_2 S'_3 S'_4).$$

The theorem follows:

Any two linear complexes set up a projective transformation between the $\{\text{spheres}\}_{\{\text{point-pairs}\}}$ of each of the circles of their common congruence.

Accordingly any circle of the congruence may be regarded as two superimposed $\{\text{pencils}\}_{\{\text{ranges}\}}$ of $\{\text{spheres}\}_{\{\text{point-pairs}\}}$, associated one with each of the complexes which define the congruence. Any circle of the congruence is common to all the complexes of the pencil of complexes

$$\sum (\lambda_1 C_1 + \lambda_2 C_2) p = 0$$

which define the congruence. Let p be any circle of the congruence. Its point-pairs and spheres are transformed by every complex $\lambda_1 : \lambda_2 = m$ of the pencil. Since the director circles of the congruence are conjugate with respect to every complex of the congruence, it follows that the $\{\text{polars}\}_{\{\text{poles}\}}$ of the $\{\text{point-pairs}\}_{\{\text{spheres}\}}$ common to p and to one of director circles, are the same with respect to every complex of the pencil. Hence these $\{\text{point-pairs}\}_{\{\text{spheres}\}}$ will be the foci of every projective transformation of the $\{\text{point-pairs}\}_{\{\text{spheres}\}}$ of p by means of any pair, m_1 and m_2 , of the complexes.

Let m_1, m_2, m_3, m_4 be any four complexes of the pencil, and P_1, P_2, P_3, P_4 be the corresponding poles of any sphere S of p , then

$$(m_1 m_2 m_3 m_4) = (P_1 P_2 P_3 P_4).$$

In particular if S_1 and S_2 be the special complexes of the pencil,

$$(m_1 S_1 m_2 S_2) = (P_1 P_{a_1} P_2 P_{a_2}) = k,$$

since P_1 and P_2 are any pair of corresponding elements and P_{a_1} and P_{a_2} are the foci. Hence the $\{\text{sphere}\}_{\{\text{point-pair}\}}$ transformations effected upon all circles com-

mon to a pair of complexes by the complexes, are formally identical. Reciprocally a transformation is effected among the complexes of a congruence by the $\left\{ \begin{smallmatrix} \text{polars} \\ \text{poles} \end{smallmatrix} \right\}$ of any $\left\{ \begin{smallmatrix} \text{point-pair} \\ \text{sphere} \end{smallmatrix} \right\}$ of a circle of the congruence. The foci are the special complexes.

53. The equation of the transformation of the point-pairs of a circle with respect to a pair of complexes assumes its simplest form when referred to the foci. We have the relation

$$(P_1 P_{d_1} P_2 P_{d_2}) = k; \text{ i. e., }^* \\ \frac{P_2 - P_1}{P_2 - P_{d_1}} : \frac{P_{d_2} - P_1}{P_{d_2} - P_{d_1}} = k.^\dagger$$

Let $P_{d_2} = \infty$, and $P_{d_1} = 0$. We have

$$\frac{P_2 - P_1}{P_2} = k,$$

or

$$P_1 = (1 - k) P_2,$$

which is the required transformation. Similarly

$$(S_1 S_{d_1} S_2 S_{d_2}) = k,$$

and

$$S_1 = (1 - k) S_2.$$

Hence the point-pair and sphere transformations effected upon the circles common to two complexes by the complexes, are formally identical.

CHAPTER VIII

THE COMPLEX OF SECOND DEGREE

A. Tangent and Polar Linear Complexes

54. The circles whose coördinates p_{ik} satisfy a homogeneous equation of the second degree form a complex of the second degree. The equation may be written

$$(126) \quad \begin{aligned} C_2 \equiv & A_{1212} p_1^2 + A_{1313} p_{13}^2 + A_{1414} p_{14}^2 + A_{2323} p_{23}^2 + A_{4242} p_{42}^2 + A_{3434} p_{34}^2 \\ & + 2A_{1213} p_{12} p_{13} + 2A_{1214} p_{12} p_{14} + 2A_{1223} p_{12} p_{23} + 2A_{1242} p_{12} p_{42} \\ & + 2A_{1234} p_{12} p_{34} + 2A_{1314} p_{13} p_{14} + 2A_{1323} p_{13} p_{23} + 2A_{1342} p_{13} p_{42} \\ & + 2A_{1334} p_{13} p_{34} + 2A_{1423} p_{14} p_{23} + 2A_{1442} p_{14} p_{42} + 2A_{1434} p_{14} p_{34} \\ & + 2A_{2342} p_{23} p_{42} + 2A_{2334} p_{23} p_{34} + 2A_{4234} p_{42} p_{34} = 0. \end{aligned}$$

* Notation of DOEHLEMANN, *Projektive Geometrie*.

† The P 's are of course to be replaced by their coördinates if a metrical interpretation is desired.

Let the polar form of C_2 be

$$(127) \quad C_2(p, p') \equiv \frac{1}{2} \sum_{ik} \frac{\partial C_2}{\partial p_{ik}} p'_{ik} \equiv \frac{1}{2} \sum \frac{\partial C_2}{\partial p'_{ik}} p_{ik}.$$

Two circles, p_{ik} and p'_{ik} , which satisfy the equation

$$(127) \quad C_2(p, p') = 0,$$

are said to be *associate* with respect to C_2 . Holding p'_{ik} fixed and letting p_{ik} vary, (127) is the equation of a complex. The theorem follows:

All the circles associate with a given circle with respect to a complex of second degree, form a linear complex.

The circle is called the *pole* of the complex, and the complex is the *polar* of the circle. It follows at once from (127):

If p'_{ik} is in the polar of p_{ik} , then p_{ik} is in the polar of p'_{ik} .

If p'_{ik} satisfies C_2 it is among the circles of its polar complex, for the polar equation reduces identically to $C_2(p')$ when

$$p_{ik} = p'_{ik}.$$

The polar complex is then said to be *tangent* to the complex of second degree.

A linear complex does not in general possess a pole with respect to a given complex of second degree. For while the six equations

$$(128) \quad \rho A'_{ik} = \frac{\partial C_2(p')}{\partial p'_{ik}},$$

define in general the ratios of the six quantities p'_{ik} uniquely, these quantities p'_{ik} do not in general satisfy the identity

$$\omega(p') = 0,$$

and hence are not the coördinates of a circle. There are in fact ∞^5 complexes, and but ∞^4 circles. Two, however, of the complexes of a congruence possess a pole. For let the congruence be defined by the pencil

$$(129) \quad \sum (\lambda_1 A'_{ik} + \lambda_2 A''_{ik}) p_{ik} = 0.$$

The seven equations

$$(130) \quad \lambda_1 A'_{ik} + \lambda_2 A''_{ik} = \frac{\partial C_2(p)}{\partial p_{ik}},$$

$$\omega(p) = 0$$

involve seven unknown quantities, p_{ik} and $\lambda_1 : \lambda_2$, and are in general capable of two solutions.

If a circle is associate with respect to two circles p'_{ik} and p''_{ik} , it will be associate with every circle of the pencil defined by these two circles. For

$$(131) \quad C_2(p, \lambda_1 p' + \lambda_2 p'') \equiv \lambda_1 C_2(p, p') + \lambda_2 C_2(p, p'') = 0.$$

B. Intersection of a Pencil of Circles with C_2

55. A pencil has two circles in common with a complex of second degree. For the equations of a pencil are

$$(132) \quad \frac{p_{12} - \lambda_1 p'_{12}}{p''_{12}} = \frac{p_{13} - \lambda_1 p'_{13}}{p''_{13}} = \dots = \frac{p_{34} - \lambda_1 p'_{34}}{p''_{34}} = \lambda_2.$$

The circles common to the pencil and C_2 are found by eliminating p_{ik} between these equations. We find

$$(133) \quad C_2(p'')\lambda_2^2 + \lambda_2\lambda_1 \sum p''_{ik} \frac{\partial C_2(p')}{\partial p'_{ik}} + \lambda_1^2 C_2(p') = 0.$$

This is a quadratic equation in $\lambda_2 : \lambda_1$, and is satisfied by two values of this ratio, i. e., the pencil and complex have two circles in common. If these two circles coincide, the pencil is said to be *tangent* to the complex. The condition that (133) has equal roots is

$$(134) \quad \left(\sum p''_{ik} \frac{\partial C_2(p')}{\partial p'_{ik}} \right)^2 - 4 C_2(p'') C_2(p') = 0.$$

If $C_2(p') = 0$, the condition becomes

$$(135) \quad \sum p''_{ik} \frac{\partial C_2(p')}{\partial p'_{ik}} = 0.$$

The theorem follows:

If p'_{ik} be a circle of a complex of second degree, the circles p''_{ik} which generate with p'_{ik} pencils tangent to the complex of second degree form a linear complex.

A similar theorem evidently holds for p''_{ik} . This complex is seen from its equation to be the tangent complex. Reciprocally: *If p'_{ik} be a circle not belonging to a complex of second degree, the circles p'_{ik} of that complex, which with p''_{ik} generate pencils tangent to the complex, belong to a linear complex polar to p''_{ik} with respect to the complex of second degree.*

56. Attention is called to the similarity between the theory here developed, and the theory of polars with respect to a conicoid. The following elements enjoy corresponding properties in this regard:

CIRCLE GEOMETRY.	POINT GEOMETRY.
Circle	Point
Pencil	Line
Linear complex	Plane
Complex of second degree	Conicoid

C. Generalization of Plücker

57. The deductions of section 54 are not strictly general. Following PLÜCKER* we may proceed as follows: Every circle whose coördinates satisfy a complex of second degree C_2 also satisfies the equation

$$C_2 + \lambda \omega(p) = 0 \quad (\lambda \text{ arbitrary}).$$

The polar form of this is

$$(136) \quad C_2(p, p') + \lambda \omega(p, p').$$

Hence to every circle there are an ∞ of polar complexes, and to every complex there is a pole. Likewise if p'_{ik} is one of the circles of C_2 , there are ∞ complexes tangent at p'_{ik} . Since $\omega(p, p') = 0$ is special, p'_{ik} is one directrix of the congruence defined by (136). The other directrix is the conjugate of p'_{ik} , and is the same with respect to all the complexes of the congruence. The second special complex is given by

$$(137) \quad \lambda = -\frac{1}{2} \frac{\frac{\partial C_2}{\partial p'_{12}} \frac{\partial C_2}{\partial p'_{34}} + \frac{\partial C_2}{\partial p'_{13}} \frac{\partial C_2}{\partial p'_{42}} + \frac{\partial C_2}{\partial p'_{14}} \frac{\partial C_2}{\partial p'_{23}}}{C_2(p')}.$$

If $C_2(p') = 0$, both directrices coincide, and

$$\lambda = \infty \quad \text{or} \quad \frac{0}{0}.$$

Circles of C_2 for which $\lambda = \frac{0}{0}$ are called singular circles of the complex. A treatment of this subject would exceed the limits of this paper. The theory of circles orthogonal to a given sphere might be pushed much further, but in showing the similarity of the theory to that of line geometry, and the development of its fundamental problems, we have accomplished our object.

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